SPATIAL AVERAGING IN THE MECHANICS OF HETEROGENEOUS AND DISPERSED SYSTEMS

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Abstract—Spatial averaging of the equations describing two single-phase media is separately considered in this paper regarding the volumes occupied by either phase with allowance for the boundary conditions on phase interfaces. The equations obtained are specialized to describe monodispersed mixtures within a "cellular" scheme. It is shown that it is necessary to consider the average values both over the whole cells and over those intersected by the boundary of a selected mixture volume.

The problem of motion in the cell is formulated. Fictitious parameters are introduced at "infinity" for the carrier phase to solve the problem. These parameters do not coincide with the average values for this phase. A closed system of equations is derived for two extreme cases: an ideal incompressible carrier fluid and an extremely viscous incompressible carrier fluid. These are correlative with the inertial and viscous motions in the cell.

Various effects are discussed in this paper. These include the radial motion of bubbles, the oriented rotation of dispersed particles (the symmetry and asymmetry of stress tensor), viscosity, phase transitions and the finite volume content of dispersed particles. Some aspects of earlier studies are critically analyzed.

The equations of continuum mechanics are averaged equations. It is possible to obtain them by successively averaging more elementary equations that describe the processes on a microscale.

The relation between the phenomenological method and the averaging method considered here is similar to the well-known one between continuum mechanics and statistical mechanics. In contrast to the pure phenomenological approach, in the averaging of microequations for macroparameters such as macrostress tensor in phases and the values that define the interactions between the phases, expressions are obtained that permit a clearer idea about their structure and potential ways of their theoretical and experimental definition.

The fundamental problem in the mathematical simulation of multiphase mixtures consists of the derivation of a *closed* system of equations describing the motion of mixture for the given physiochemical properties of each phase individually and for the given initial structure of mixture. Without the use of the additional empirical relationships and coefficients this problem can have a sufficiently accurate and substantial (for example, by averaging more elementary equations) solution only for very particular classes of heterogeneous media and processes. Nevertheless, these cases are of great methodical significance, because the equations that correspond to these cases can be regarded as ultimate ones or standards that give the "base points" for less accurate simulation of complex real mixtures by using the auciliary hypotheses and empirical or phenomenological relationships. Two such ultimate cases are considered in sections 11 and 12.

In heterogeneous systems unlike homogeneous and colloid ones, the dimensions of nonuniformities and impurities are far greater than the intermolecular distances. Therefore, the parameters and equations that describe micromotions, henceforth referred to as microparameters and microequations, are the well known parameters and equations of the motion of several (depending on a number of phases) single-phase continua. Below, to simplify the discussion, the case of a two-phase mixture is considered.

1. THE EQUATIONS DESCRIBING MICROMOTION AND MICROPROCESSES WITHIN THE PHASES AND AT THE PHASE INTERFACES

Keeping in mind the abovementioned, we shall assume that in the points occupied by the ith phase, i.e. within a volume V_i, the following microequations are valid, which may be expressed in a unified form:

$$r' \in V_i$$

$$\rho_i^{0\prime} \frac{d_i' e_i'}{dt} = \nabla'^k \psi_i'^k + \rho_i^{0\prime} F_i' \qquad [1.1]$$

$$\left(\frac{d_i' e_i'}{dt} = \frac{\partial e_i'}{\partial t} + v_i'^k \nabla'^k e_i' \quad \text{or} \quad \rho_i^{0\prime} \frac{d_i' e_i'}{dt} = \frac{\partial \rho_i^{0\prime} e_i'}{\partial t} + \nabla'^k \rho_i^{0\prime} e_i' v_i'^k \right) \qquad [1.2]$$

$$
\begin{aligned}
e_i' &= 1, & v_i', & \quad u_i' + 1/2(v_i')^2, & [r' \times v_i'] \\
\psi_i'^k &= 0, & \sigma_i'^k, & \quad \sigma_i'^k \cdot v_i' - q_i'^k, & [r' \times \sigma_i'^k] \\
F_i' &= 0, & g_i', & \quad g_i' \cdot v_i', & [r' \times g_i'].
\end{aligned}
\qquad [1.3]
$$

Here and throughout summation is done only with respect to the upper (coordinate) indices; r' is a radius-vector between the fixed point 0 and the point, in question occupied by the ith phase; $\rho_i^{0\prime}$, v_i', $\sigma_i'^{kl}$, g_i', q_i' and u_i are, respectively, the instant values of density, velocity, stress tensor, specific external mass force, heat flux vector and internal energy (the microparameters of which will be denoted by primes), which are the averages within the volumes $d'V \ll a^3$ and times $dt \ll t_a$, where a and t_a are the characteristic dimension of nonuniformities and characteristic time, respectively. We shall consider the case where the phase materials are nonpolar media and when there is no effect of magnetic field, i.e. when the microstress tensor is symmetric

$$\sigma_i'^{kl} = -p_i' \delta^{kl} + \tau_i'^{kl}, \qquad \sigma_i'^{kl} = \sigma_i'^{lk} \qquad [1.4]$$

and the equation of the moment of impulse relative to some point 0 is a consequence of the momentum equation. From the energy equation and momentum equation an equation of heat flux along the trajectory of microparticles follows

$$\rho_i^{0\prime} \frac{d_i' u_i'}{dt} = \frac{p_i' d_i \rho_i^{0\prime}}{\rho_i^{0\prime} dt} + \tau_i'^{kl} \nabla'^k v_i'^l - \nabla'^k q_i'^k. \qquad [1.5]$$

The boundary (or jump) conditions that represent the phase interaction can be given at the phase interface S_{12} as follows:

$$r' \in S_{12}$$

$$\xi_2' = -\xi_1' \quad (\xi_i' = -\rho_i^{0\prime}(v_i' \cdot n_i' - N' \cdot n_i'), \quad n_1' = -n_2') \qquad [1.6]$$

$$\sigma_1'^k n_1'^k + \xi_1' v_1' = \sigma_2'^k n_2'^k + \xi_2' v_2' \qquad [1.7]$$

$$\sigma_1'^k n_1'^k \cdot v_1' - q_1^k n_1'^k + \xi_1'[u_1' + 1/2(v_1')^2] = \sigma_2'^k n_2'^k \cdot v_2' - q_2'^k n_2'^k + \xi_2'[u_2' + 1/2(v_2')^2] \qquad [1.8]$$

where n_i' is external, with respect to the ith phase, normal to the interface S_{12}; $N' \cdot n_i'$ is velocity of the displacement of the surface S_{12} along n_i'; ξ_i' is mass flux due to phase transitions into the

*i*th phase across the phase interface unit. For the sake of simplicity, capillary effects are disregarded here. Jump conditions that take into account the surface tension can be found in the papers of Drew (1971) and Delhaye (1974). The generalization of the results of this paper with allowance for the surface tension is given in the book of Nigmatulin (1978).

In analogy with [1.1], the equations of conservation [1.6]–[1.8] can be rewritten in a unified form:

$$\psi_1'^k n_1'^k + \xi_1' e_1' = \psi_2'^k n_2'^k + \xi_2' e_2' \qquad [1.9]$$

where $\psi_i'^k$ and e_i' are given in [1.3].

To close the system of microequations it is necessary to use the equations that define the physical properties of the phase materials.

2. THE AVERAGED VALUES AND MAIN ASSUMPTIONS

The difference in the methods of averaging (spatial, time, spatial-time, statistical) is of no significance for deriving a system of the averaged equations. It reveals importance the selection of basic hypotheses, the elaboration of the methods for the theoretical and experimental determination of the average values and relations between them. The general questions of averaging in application to two-phase mixtures are discussed in a number of works (Landau & Lifshitz 1959; Vernier & Delhaye 1968; Slatery 1969; Batchelor 1970b; Brenner 1970a; Drew 1971; Drew & Segel 1971; Staffman 1971; Whitaker 1973; Buevich & Markov 1973; Ishii 1975; Buevich & Shchelchkova 1976; Nigmatulin 1978).

To proceed to the averaged variables and equations, we introduce an elementary macro-volume dV confined within a surface dS and an elementary plane macrosurface ds, the characteristic linear dimensions of which dx are many times greater than the nonuniformities a (diameters of drops, bubbles, particles, pores, distances between them, film thicknesses, etc.), but at the same time much less than the characteristic macrodimension L of a problem (length and diameter of a nozzle, pipe, the characteristic disturbance attenuation distance, its length and so on)

$$L \gg dx \gg a \gg d'x, \qquad dV \sim (dx)^3, \qquad ds \sim (dx)^2. \qquad [2.1]$$

Further on, two basic assumptions will be used:

(1) It is always possible to single out in the region of the mixture motion such small, but representative volumes dV and surfaces ds about any point defined by the radius-vector **r**. We designate elements of volume dV and cross-section ds belonging to the *i*th phase ($i = 1, 2$), as dV_i and ds_i, respectively

$$dV_1 + dV_2 = dV, \qquad ds_1 + ds_2 = ds, \qquad (2.2]$$

and the interface inside dV as dS_{12}. The volume α_{Vi} and surface α_{Si} phase concentrations, the phase interface s_{12} in the volume unit, as well as the averaged or macroscopic values introduced by integration of the volume averaging of phases dV_i, phase cross-sections ds_i and phase interfaces dS_{12} can be correlated with the center of a volume dV or a surface ds defined by the vector **r**:

$$\alpha_{Vi} = \frac{dV_i}{dV}, \qquad \alpha_{Si} = \frac{ds_i}{ds}, \qquad s_{12} = \frac{dS_{12}}{dV}, \qquad [2.3]$$

$$(\alpha_{V1} + \alpha_{V2} = 1, \quad \alpha_{S1} + \alpha_{S2} = 1),$$

$$\langle\varphi_i'\rangle_{Vi} = \frac{1}{\mathrm{d}V_i}\int_{\mathrm{d}V_i}\varphi_i'\,\mathrm{d}'V, \quad \langle\varphi_i'\rangle_{Si} = \frac{1}{\mathrm{d}s_i}\int_{\mathrm{d}s_i}\varphi_i'\,\mathrm{d}'s \qquad [2.4]$$

$$\langle\varphi_i'\rangle_{12} = \frac{1}{\mathrm{d}S_{12}}\int_{\mathrm{d}S_{12}}\varphi_i'\,\mathrm{d}'s. \qquad [2.5]$$

(2) The values averaged in this manner are smoothly changing when r varies at distances of the order of L, i.e. at these distances the functions

$$\frac{\partial\langle\varphi_i'\rangle_{Vi}}{\partial x} \sim \frac{\partial\langle\varphi_i'\rangle_{Si}}{\partial x} \sim \frac{\partial\langle\varphi_i'\rangle_{12}}{\partial x} \sim \frac{\varphi_0}{L} \qquad [2.6]$$

are regular.

From the smoothness condition [2.6] for surface-average values, the coincidence of surface-average and volume-average values follows

$$\alpha_{Vi} = \alpha_{Si} = \alpha_i, \quad \langle\varphi_i'\rangle_{Vi} = \langle\varphi_i'\rangle_{Si} = \langle\varphi_i'\rangle_i. \qquad [2.7]$$

It results from the fact that a volume-average value can be presented as an integral with respect to the linear coordinate x, normal to ds from the surface-average value

$$\langle\varphi_i'\rangle_{Vi} = \frac{1}{\mathrm{d}V}\int_{x-\mathrm{d}x}^{x+\mathrm{d}x}\langle\varphi_i'(x')\rangle_{Si}\,\mathrm{d}s_i(x')\,\mathrm{d}'x = \langle\varphi_i'\rangle_{Si} + \varphi_0 0\Big(\frac{\mathrm{d}x}{L}\Big) \approx \langle\varphi_i'\rangle_{Si}. \qquad [2.8]$$

Spatial phase averaging and, consequently, by virtue of [2.7], phase averaging in general, does not change the tensor character and rank of the average values, namely: a scalar remains a scalar, a vector a vector, a second order tensor a second order tensor, and the property of symmetry or asymmetry is retained. These assertions may be easily proved.

In Nikolaevsky's work (1975) averaged equations are suggested to be derived, in which the surface-average and volume-average values [2.4] would differ, and it is asserted that these equations are more general and, in particular, will contain the asymmetric stress tensor $\langle\sigma'\rangle_S^{kl}$. This conception is incorrect, since from the above given considerations it follows that, if the surface-average values $\langle\varphi'\rangle_S$ oscillating at distances $\sim\mathrm{d}x$ are not allowed, then the difference between the surface-average and volume-average values is possible to obtain solely by introducing the surface-average values $\langle\varphi'\rangle_{S_*}$ that differ from [2.4], especially by averaging not on an arbitrary plane macrosurface ds, but on the one separated with respect to nonuniformities or impurities in the mixture, adjacent to ds and differing from ds by $\delta r' \sim a \lesssim \mathrm{d}x$. At the same time, a method of the introduction of such surfaces ds_* and, correspondingly, volumes dV_* depends on the mixture structure and must be specified, so that it can, in principle, be used to determine the relation between $\langle\varphi'\rangle_{S_*}$, $\langle\varphi'\rangle_{V_*}$ and the average values [2.4]. It should be remembered that $\langle\varphi'\rangle_{S_*} \neq \langle\varphi'\rangle_S$ which may result in $\langle\varphi'\rangle_{S_*} \neq \langle\varphi'\rangle_V$, but, nevertheless, the equality $\langle\varphi'\rangle_S = \langle\varphi'\rangle_V$ is true too. Therefore, such an analysis with the introduction of ds_* has only methodic distinctions when realized correctly. Whereas the great generality of equations, in which the surface-average and volume average values differ, is illusory even with the introduction of ds_*. Below, in sections 7 and 8 ds_1 + ds_{21S} is used as ds_* (see the discussion following [8.7]), it is possible to use ds_{1V} too (see section 7).

3. AVERAGING OF CONSERVATION EQUATIONS

If the differential conservation equations [1.1] describing the mixture micromotions are integrated over the elementary macrovolume dV_i occupied by the ith phase, and the equations at the interface [1.9] are integrated over dS_{12}, then, by taking into consideration the average

values [2.4], [2.5], we obtain the averaged equations, which contain the average values from the microparameter derivatives with respect to time and coordinates

$$\left\langle \frac{\partial}{\partial t}\rho_i^{0\prime}e_i^\prime \right\rangle_i + \langle \nabla^{\prime k}\rho_i^{0\prime}e_i^\prime v_i^{\prime k}\rangle_i = \langle \nabla^{\prime k}\psi_i^{\prime k}\rangle_i + \langle \rho_i^{0\prime}F_i^\prime\rangle_i$$ [3.1]

$$\left(\left\langle \frac{\partial}{\partial t}\rho_i^{0\prime}e_i^\prime \right\rangle_i + \langle \nabla^{\prime k}\rho_i^{0\prime}e_i^\prime v_i^{\prime k}\rangle_i = \left\langle \rho_i^{0\prime}\frac{d_i^\prime e_i^\prime}{dt} \right\rangle_i\right)$$ [3.1a]

$$\langle \psi_1^{\prime k}n_1^{\prime k}\rangle_{12} + \langle \xi_1^\prime e_1^\prime\rangle_{12} = \langle \psi_2^{\prime k}n_2^{\prime k}\rangle_{12} + \langle \xi_2^\prime e_2^\prime\rangle_{12}.$$ [3.2]

Averaging of derivatives, as distinct from that in a single-phase medium or over the whole volume of mixture (Batchelor 1970), in averaging over phases it is impossible to assume that the average of a derivative is equal to the derivative of the average. The averaging over phases and differentiation with respect to time or coordinates are noncommutative. This is testified by a simple one-dimensional example for a laminated system shown in figure 1, where the second phase is shaded. It is evident that

$$\left\langle \frac{\partial \varphi_1^\prime}{\partial^\prime x} \right\rangle_1 < 0, \quad \frac{\partial \langle \varphi_1^\prime\rangle_1}{\partial x} > 0 \quad \text{or} \quad \left\langle \frac{\partial \varphi_1^\prime}{\partial^\prime x} \right\rangle_1 \neq \frac{\partial \langle \varphi_1^\prime\rangle_1}{\partial x}.$$

For any differentiable scalar, vector or tensor function $\varphi_i^\prime$ and for an elementary macro-volume $dV = dV_1 + dV_2 = $ const fixed in space and bounded by a surface $dS = dS_1(t) + dS_2(t) = $ const, the following equality is valid

$$\frac{d}{dt}\int_{dV_i(t)} \varphi_i^\prime \, dV = \int_{dV_i} \frac{\partial \varphi_i^\prime}{\partial t} d^\prime V + \int_{dS_i + dS_{12}} \varphi_i^\prime N_i^{\prime n} \, d^\prime s.$$ [3.3]

As $dS_i(t)$ transfers along the stationary surface dS, then on $dS_i(t)$ the normal velocity of this surface displacement $N_i^{\prime n} = 0$. Considering the average value definition [2.3]–[2.5] and the property [2.7] and also regarding $\varphi_i^\prime = 1$, we shall have

$$\frac{\partial}{\partial t}\alpha_i\langle \varphi_i^\prime\rangle_i = \alpha_i\left\langle \frac{\partial \varphi_i^\prime}{\partial t} \right\rangle_i + s_{12}\langle \varphi_i^\prime N^\prime \cdot n_i^\prime\rangle_{12}$$ [3.4]

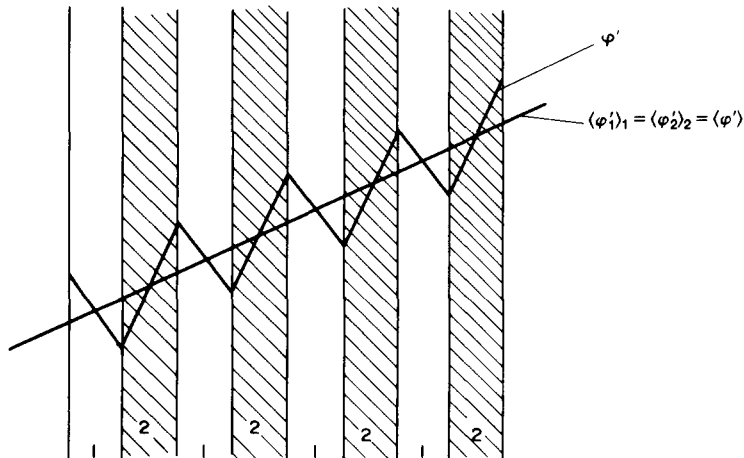$$\frac{\partial \alpha_i}{\partial t} = s_{12}\langle N^\prime \cdot n_i^\prime\rangle_{12}.$$ [3.5]

Figure 1.

The left side comprises partial derivatives with respect to time due to immobility of the center of volume dV, to which the average values are related. As a result, we obtain the formula

$$\alpha_i \left\langle \frac{\partial \varphi_i'}{\partial t} \right\rangle_i = \alpha_i \frac{\partial \langle \varphi_i' \rangle_i}{\partial t} - s_{12}(\langle \varphi_i' N' \cdot n_i' \rangle_{12} - \langle \varphi_i' \rangle_i \langle N' \cdot n_i' \rangle_{12}).$$ [3.6]

For any differentiable vector or tensor function $\varphi_i'^n = \varphi_i'^k n_i'^k$, an equality is valid

$$\int_{V_i} \nabla'^k \varphi_i'^l \, d'V = \int_{S_i} \varphi_i'^l n_i'^k \, d's + \int_{S_{12}} \varphi_i'^l n_i'^k \, d's$$ [3.7]

resulting from the Gauss–Ostrogradsky theorem as applied to the volume V_i. Expressing the member integrals in the form of averaged functions, we get

$$\int_V \alpha_i \langle \nabla'^k \varphi_i'^l \rangle_i \, dV = \int_S \alpha_i \langle \varphi_i'^l \rangle_i n_i^k \, ds + \int_V s_{12} \langle \varphi_i'^l n_i^k \rangle_{12} \, dV.$$

We shall now transform the surface integral into a volume one and, with allowance for that this equation is valid for an arbitrary macrovolume V, we obtain the formula

$$\alpha_i \langle \nabla'^k \varphi_i'^l \rangle_i = \nabla^k \alpha_i \langle \varphi_i'^l \rangle_i + s_{12} \langle \varphi_i'^l n_i'^k \rangle_{12}.$$ [3.8]

The formulas, close to [3.6] and [3.8] for averaging the derivatives, have earlier been considered by Saffman (1971), Slatery (1969) and Whitaker (1973).

The averaged phase mass equation

Formulas [3.6] and [3.8] give precise (within the limits of the hypotheses accepted) expressions for the derivatives with respect to time and coordinates averaged over the phases. In view of these formulas, [3.1] and [3.2] give in the case of $e_i' = 1$, $\psi_i' = 0$ (see [1.3], [1.6]) the averaged phase mass equation in the form

$$\frac{\partial \rho_i^0 \alpha_i}{\partial t} + \nabla^k \rho_i^0 \alpha_i v_i^k = J_{ji}; \quad i, j = 1,2; \quad i \ne j$$ [3.9]

$$\rho_i^0 = \langle \rho_i^{0\prime} \rangle_i, \quad \rho_i^0 v_i = \langle \rho_i^{0\prime} v_i' \rangle_i$$ [3.10]

$$J_{21} = -J_{12} = s_{12} \langle \xi_1' \rangle_{12} = -s_{12} \langle \xi_2' \rangle_{12}.$$ [3.11]

Averaging of substantial derivatives along the trajectories of microparticles

From [3.1a] by using [3.6], [3.8] and [3.9], it is possible to obtain a formula relating the mass-average values of the substantial derivatives with respect to time along the microparticle trajectories, to the average parameter values and their derivatives, in particular, to the substantial derivative with respect to time along the averaged trajectory, or that of an elementary microparticle mass center path

$$\alpha_i \left\langle \rho_i^{0\prime} \frac{d_i e_i}{dt} \right\rangle_i = \rho_i \alpha_i \frac{d_i e_i}{dt} + \nabla^k \alpha_i \langle \rho_i^{0\prime} \Delta e_i' \Delta v_i'^k \rangle_i + J_{ji}(e_{(12)i} - e_i)$$ [3.12]

$$\rho_i^0 e_i = \langle \rho_i^{0\prime} e_i' \rangle_i, \quad \Delta e_i' = e_i' - e_i, \quad \Delta v_i' = v_i' - v_i$$ [3.13]

$$J_{ji}e_{(12)i} = s_{12}\langle \xi_i' e_i' \rangle_{12}$$ [3.14]

$$\frac{d_i e_i}{dt} = \frac{\partial e_i}{\partial t} + v_i^k \nabla^k e_i.$$ [3.15]

The general form of the conservation equations

After using [3.8] for $\varphi_i'' = \psi_i''(l = k)$ and [3.12], the averaged [3.1] and [3.2] for the conservation of momentum, energy and moment of phase impulses can be expressed in terms of the averaged functions and their derivatives with respect to time and coordinates

$$\rho_i \frac{d_i e_i}{dt} = \nabla^k \psi_i^k + R_{ji}^{(\psi)} + J_{ji}(e_{(12)i} - e_i) + \rho_i F_i$$ [3.16]

$$(i, j = 1, 2; \ i \neq j)$$

$$R_{12}^{(\psi)} + R_{21}^{(\psi)} + J_{12}e_{(12)2} + J_{21}e_{(12)1} = 0$$ [3.17]

$$\rho_i = \rho_i^0 \alpha_i$$ [3.18]

$$\rho_i^0 F_i = \langle \rho_i^{0'} F_i' \rangle_i$$ [3.19]

$$\psi_i^k = \alpha_i \langle \psi_i'^k \rangle_i - \alpha_i \langle \rho_i^{0'} \Delta e_i' \Delta v_i'^k \rangle_i$$ [3.20]

$$R_{ji}^{(\psi)} = s_{12} \langle \psi_i'^k n_i'^k \rangle_{12}$$ [3.21]

where the values written in the second, third and fourth columns in [1.3] are taken as the microparameters e_i', $\psi_i'^k$ and F_i'.

4. THE AVERAGED PHASE MOMENTUM EQUATIONS

Let us consider the momentum equation for the ith phase written in the form of the averaged values, following from [3.16] to [3.21] for the second column of the microparameter values in [1.3]

$$\rho_i \frac{d_i v_i}{dt} = \nabla^k \sigma_i^k + R_{ji} + J_{ji}(v_{(12)i} - v_i) + \rho_i g_i$$ [4.1]

$$(i, j = 1, 2; \ i \neq j)$$

$$R_{12} + R_{21} = J_{12}(v_{(12)1} - v_{(12)2})$$ [4.2]

$$\sigma_i^{kl} = \alpha_i \langle \sigma_i'^{kl} \rangle_i + \Omega_i^{kl}, \quad \Omega_i^{kl} = -\alpha_i \langle \rho_i^{0'} \Delta v_i'^k \Delta v_i'^l \rangle_i$$ [4.3]

$$J_{ji}v_{(12)i} = s_{12} \langle \xi_i' v_i' \rangle_{12}$$ [4.4]

$$R_{ji} = s_{12} \langle \sigma_i'^k n_i'^k \rangle_{12},$$ [4.5]

$$\rho_i^0 g_i = \langle \rho_i^{0'} g_i' \rangle_i.$$ [4.6]

The tensor σ_i^{kl} determines the action upon the ith phase along the surface boundary dS_i of a selected mixture volume, where the addend Ω_i^{kl} determines a fluctuating momentum transfer or

fluctuating stresses. This component is similar to the Reynolds stresses in the turbulent motion of fluid. There is a wide class of dispersed system motions for which these components may be neglected. The phenomenological equations (see, e.g. Nigmatulin 1970 and review of Kraiko *et al.* 1972) for the dispersed mixtures correspond to those motions. Interphase momentum exchange occurs due to the interphase force R_{ji} and to phase transitions $J_{ji}v_{(12)i}$; the addend $J_{ji}(v_{(12)i} - v_i)$ being regarded as a jet force.

The tensor of microstresses in every point of the ith phase can be presented in the form

$$\sigma_i'^{kl} = -p_i'\delta^{kl} + \tau_i'^{kl} = -(p_i + \Delta p_i')\delta^{kl} + \tau_i'^{kl} \quad (\tau_i'^{kk} = 0) \tag{4.7}$$

$$p_i = \langle p_i' \rangle_i \tag{4.8}$$

where p_i is average pressure in the ith phase, $\Delta p_i'$ is pressure fluctuation, $\tau_i'^{kl}$ is the deviator of microstresses. The following accurate formula can be proved (see Nigmatulin 1978):

$$-s_{12}\langle p_i n_i' \rangle_{12} = -\frac{1}{dV}\int_{dS_{12}} p_i n_i' \, d's = p_i \nabla \alpha_i. \tag{4.9}$$

This expression defines the component of interphase force due to the ith phase stream tube expansion. This force was first noticed by Rakhmatulin (1956). It was also distinguished by Ishii (1975). Therefore, the interphase force R_{ji} can be represented in the form

$$R_{ji} = p_i \nabla \alpha_i + \Delta R_{ji}^{(p)} + R_{ji}^{(\tau)} \tag{4.10}$$

$$\Delta R_{ji}^{(p)} = -s_{12}\langle \Delta p_i' n_i' \rangle_{12}, \quad R_{ji}^{(\tau)} = s_{12}\langle \tau_i'^k n_i'^k \rangle_{12} \tag{4.11}$$

where $\Delta R_{ji}^{(p)}$ is the interphase force due to small-scale pressure fluctuations (it contains the so-called virtual mass force, the Magnus–Zukovsky force due to rotation, etc.), $R_{ji}^{(\tau)}$ is interphase force due to shear stresses.

5. THE AVERAGED PHASE ENERGY EQUATIONS

Let us consider the averaged conservation energy equation of the ith phase. For this purpose in [3.16], [3.17] and the determining relations [3.18]–[3.21] the values e_i', $\psi_i'^k$ and F_i' are taken from the third column of [1.3]. The averaged value of the ith phase total energy is thus expanded into three components: the internal energy u_i, the kinetic energy of macromotion $1/2\, v_i^2$ and the kinetic energy of fluctuating motion k_i:

$$\langle \rho_i^{0'} e_i' \rangle_i = \langle \rho_i^{0'}(u_i' + 1/2\, v_i'^2) \rangle_i = \rho_i^0 u_i + 1/2\rho_i^0 v_i^2 + \rho_i^0 k_i \tag{5.1}$$

$$\rho_i^0 u_i = \langle \rho_i^0 u_i' \rangle_i \tag{5.2}$$

$$\rho_i^0 k_i = 1/2\langle \rho_i^{0'} v_i'^2 \rangle_i - 1/2\rho_i^0 v_i^2 = 1/2\langle \rho_i^{0'}(\Delta v_i')^2 \rangle_i. \tag{5.3}$$

The averaged interphase energy flux $E_{ji} = R_{ji}^{(\psi)} + J_{ji}e_{(12)i}$ into the ith phase at the surface S_{12} can be represented in the form of the sum of three components:

$$E_{ji} = W_{ji} + Q_{ji} + J_{ji}(u_{(12)i} + 1/2v_{(12)i}^2 + 1/2(\delta v_i)^2) \tag{5.4}$$

$$W_{ji} = s_{12}\langle \sigma_i'^{kl} n_i'^k v_i'^l \rangle_{12} \tag{5.5}$$

$$Q_{ji} = -s_{12}\langle q_i'^k n_i'^k \rangle_{12} \tag{5.6}$$

$$J_{ji}u_{(12)i} = s_{12}\langle \xi_i' u_i' \rangle_{12} \qquad [5.7]$$

$$J_{ji}(\delta v_i)^2 = \langle \xi_i'(v_i' - v_{(12)i})^2 \rangle_{12} \qquad [5.8]$$

where W_{ji} defines interphase exahange by energy due to the work of interphase forces, Q_{ji}—due to interphase heat transfer, and the latter addend in [5.4]—due to phase transitions.

We shall introduce the values c_i^k and q_i^k defining the averaged energy exchange in the ith phase along the external boundaries dS_i of the volume dV_i:

$$c_i^k = \alpha_i[\langle \sigma_i'^k \cdot v_i' \rangle_i - 1/2\langle \rho_i^{0'}(v_i')^2 \Delta v_i'^k \rangle_i]$$

$$= \sigma_i^k \cdot v_i + \alpha_i[\langle \sigma_i'^k \cdot \Delta v_i' \rangle_i - 1/2\langle \rho_i^{0'}(\Delta v_i')^2 \Delta v_i'^k \rangle_i] \qquad [5.9]$$

$$q_i^k = \alpha_i \langle q_i'^k \rangle_i + \Gamma_i^k, \qquad \Gamma_i^k = -\alpha_i \langle \rho_i^{0'} \Delta u_i' \Delta v_i'^k \rangle_i \qquad [5.10]$$

where c_i^k is mechanical energy transfer due to the work of surface forces and fluctuating motion, q_i^k is the averaged heat flux, a portion of which Γ_i^k is transferred due to the fluctuating motion.

As a result, the ith phase averaged energy equation that results from [3.16] can be presented in the form:

$$\rho_i \frac{d_i}{dt}\left(u_i + k_i + \frac{v_i^2}{2}\right) = \nabla^k(c_i^k - q_i^k) + \rho_i g_i \cdot v_i + \rho_i H_i$$

$$+ W_{ji} + Q_{ji} + J_{ji}\left[u_{(12)i} - u_i + \frac{v_{(12)i}^2 - v_i^2}{2} + \frac{(\delta v_i)^2}{2} - k_i\right] \qquad [5.11]$$

$$\rho_i^0 H_i = \langle \rho_i^{0'} g_i' \cdot v_i' \rangle - \rho_i^0 g_i \cdot v_i = \langle \rho_i^{0'} g_i' \cdot \Delta v_i' \rangle_i. \qquad [5.12]$$

An equation of energy conservation at the interphase boundary results from [3.17]

$$W_{21} + W_{12} + Q_{21} + Q_{12} + J_{12}\left[u_{(12)2} - u_{(12)1} + \frac{v_{(12)2}^2 - v_{(12)1}^2}{2} + \frac{(\delta v_2)^2 - (\delta v_1)^2}{2}\right] = 0. \qquad [5.13]$$

The kinetic energy equation of macroscopic motion follows from [4.1]

$$\rho_i \frac{d}{dt}\left(\frac{v_i^2}{2}\right) = v_i \cdot \nabla^k \sigma_i^k + R_{ji} \cdot v_i + J_{ji}(v_{(12)i} - v_i) \cdot v_i + \rho_i g_i \cdot v_i. \qquad [5.14]$$

The equation for internal energy or the averaged heat influx equation of the ith phase is obtained by averaging [1.5] with allowance for [3.8] and [3.12]

$$\rho_i \frac{d_i u_i}{dt} = -\nabla^k q_i^k + Q_{ji} + J_{ji}(u_{(12)i} - u_i) + \rho_i A_i \qquad [5.15a]$$

$$\rho_i^0 A_i = \langle \sigma_i'^{kl} \nabla'^k v_i'^l \rangle_i \qquad [5.15b]$$

where A_i corresponds to the average power of the work of internal forces in the ith phase. After subtraction of [5.14] and [5.15] from [5.11] we shall obtain the fluctuating motion kinetic energy equation

$$\rho_i \frac{d_i k_i}{dt} = \nabla^k \Lambda_i^k + \sigma_i^{kl} \nabla^k v_i^l - \rho_i A_i + (W_{ji} - R_{ji} \cdot v_i) + \rho_i H_i + J_{ji}(k_{(12)i} - k_i) \qquad [5.16]$$

$$\Lambda_i^k = -\alpha_i[1/2\langle\rho_i^{0\prime}(\Delta v_i')^2\Delta v_i'^k\rangle_i - \langle\sigma_i'^k \cdot \Delta v_i'\rangle_i]$$ [5.17]

$$J_{ji}k_{(12)i} = 1/2J_{ji}(v_{(12)i} - v_i)^2 + 1/2\langle\xi_i'(v_i' - v_{(12)i})^2\rangle_{12}$$ [5.18]

where Λ_i^k is the flux of fluctuating energy due to fluctuating motion.

6. THE AVERAGED EQUATION OF THE MOMENTUM OF FLUCTUATING MOTION

Besides kinetic energy, the fluctuating motion $\Delta v_i'$ is characterized by the momentum M_i

$$\rho_i^0 M_i = \langle\rho_i^{0\prime}[\Delta r' \times \Delta v_i']\rangle_i.$$ [6.1]

Then the averaged moment of momentum is equal to

$$\langle\rho_i^{0\prime}[r' \times v_i']\rangle_i = \rho_i^0[r \times v_i] + \rho_i^0 M_i.$$ [6.2]

The equation of the fluctuating motion momentum is obtained in a similar manner as [5.16]. In case of the absence of phase transitions ($\xi_i' = 0$) and small fluctuations of density ($\Delta\rho_i^{0\prime} \ll \rho_i^0$), it takes the form

$$\rho_i\frac{d_iM_i}{dt} = \nabla^k\mu_i^k + D_{ji} + \rho_iL_i \quad (D_{12} + D_{21} = 0)$$ [6.3]

$$\mu_i^k = \alpha_i\langle[\Delta r' \times \Delta\sigma_i'^k]\rangle_i - \rho_i^0\alpha_i\langle[\Delta r' \times v_i']\Delta v_i'^k\rangle_i$$ [6.4]

$$D_{ji} = s_{12}\langle[\Delta r' \times \sigma_i'^k n_i'^k]\rangle_{12}.$$ [6.5]

In the case when the oriented microscale phase rotation is absent, it is fulfilled identically and has not to be used in the analysis.

Unlike the phenomenological approach, the averaging method made it possible to successively take into consideration the influence of fluctuating phase motion and to obtain expressions for the determination (in terms of the distribution of microparameters) of such macroscopic characteristics, as the stress tensor in phases, the intensity of interphase interactions, and energy fluxes of the various form. Realization of these expressions leading to rheological relations now only between the macroparameters (which can be called the explicit rheological relations) and, as a result, to the closure of the system of equations, has to be performed with due regard to the structure and physical properties of phases in the mixture. This is the basic problem in simulating the heterogeneous media.

7. DISPERSED MIXTURES AND THEIR INTERPHASE INTERACTIONS

Let us consider a two-phase mixture, where one phase referred to as the dispersed phase comprises drops, bubbles or solid particles (it will be regarded as the second phase), and which occupies a volume V_2, equal to the sum of different volumes occupied by various dispersed particles. The remaining volume V_1 is occupied by the dispersive or carrier phase (it will be assumed as the first phase).

Any microvolume dV in the dispersed mixture may be at any instant subdivided into dN cells (according to the number of dispersed particles), each having the form of, approximately, a polyhedron with one dispersed particle in its center. A linear dimension of the cell is equal to the average distance between the impurities $2l$. In a representative macrovolume, the number of dispersed particles and therefore the number of cells is great. At the same time, a portion of

cells dN_V in that number that occupies the volume dV_V and includes the particle surface dS_{12V} lies entirely within the selected macrovolume dV, while a portion of them dN_S in that number that occupies the volume dV_S and includes the interface dS_{12S} is intersected by the boundary dS. The volume dV_V is bounded by the surface dS_{1V} passing in the vicinity of dS, but entirely within the carrier phase (see figure 2). Apart from the averages over the "whole" cells, it is necessary to consider in the general cases the average values over the intersected cells

$$n = \frac{dN}{dV} \approx \frac{dN_V}{dV}, \quad s_{12V} = \frac{dS_{12V}}{dV} \approx s_{12}, \quad s_{12S} = \frac{dS_{12S}}{dV} \tag{7.1}$$

$$\langle \varphi_i' \rangle_{iV} = \frac{1}{dV_{iV}} \int_{dV_{iV}} \varphi_i' \, d'V, \quad \langle \varphi_i' \rangle_{iS} = \frac{1}{dV_{iS}} \int_{dV_{iS}} \varphi_i' \, d'V \tag{7.2}$$

$$\langle \varphi_i' \rangle_{12V} = \frac{1}{dS_{12V}} \int_{dS_{12V}} \varphi_i' \, d's, \quad \langle \varphi_i' \rangle_{12S} = \frac{1}{dS_{12S}} \int_{dS_{12S}} \varphi_i' \, d's \tag{7.3}$$

$$\langle \varphi_i' \rangle_i = \langle \varphi_i' \rangle_{iV} + \frac{dV_{iS}}{dV} \langle \varphi_i' \rangle_{iS} \tag{7.4}$$

$$s_{12} \langle \varphi_i' \rangle_{12} = s_{12V} \langle \varphi_i' \rangle_{12V} + s_{12S} \langle \varphi_i' \rangle_{12S}. \tag{7.5}$$

By virtue of a small volume and number of the intersected cells $(dN_S \ll dN,\ s_{12S} \ll s_{12V},\ dV_{iS} \ll dV_{iV})$, the second addends in [7.4] and [7.5] can contribute to the averaged values only when microparameter fluctuations are many times greater than the average value in the cell. It is realized when averaging the highly varying space derivatives of the physical parameters and also when averaging over the interphase boundaries. To illustrate this, let us consider the integral with respect to the vector or tensor field divergence ψ_2' over the volume dV_{2S} contained within the surface $dS_2 + dS_{12S}$. Taking into consideration the Gauss–Ostrogradsky theorem and [7.2], [7.3], we obtain

$$\frac{dV_{2S}}{dV} \langle \nabla'^k \psi_2'^k \rangle_{2S} - s_{12S} \langle \psi_2'^k n_2'^k \rangle_{12S} = \nabla^k \alpha_2 \langle \psi_2'^k \rangle_2. \tag{7.6}$$
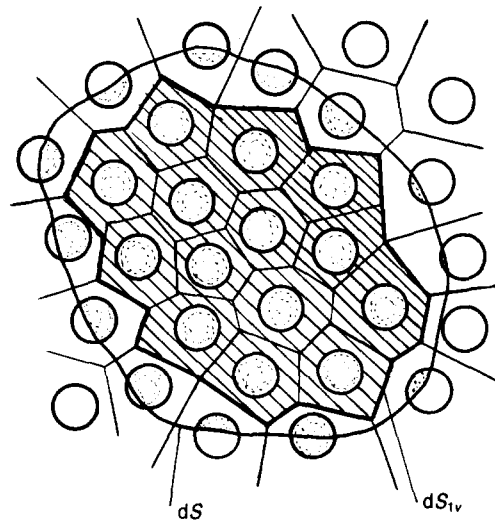

Figure 2.

The addend on the right side is determined by the gradients of macroparameters. It is clear that with this values (see [3.16]) it is necessary to take into account at least one of the discussed addends on the left side related to the averaging over the intersected cells.

In application to the averaged intensities of phase interactions [3.21] the separation of [7.5] leads to the following form

$$R_{ji}^{(\psi)} = R_{jiV}^{(\psi)} + R_{jiS}^{(\psi)}$$ [7.7]

$$R_{jiV}^{(\psi)} = s_{12V}\langle\psi_i'^k n_i'^k\rangle_{12V}, \quad R_{jiS}^{(\psi)} = s_{12S}\langle\psi_i'^k n_i'^k\rangle_{12S}$$ [7.8]

$$\begin{align} \psi_i'^k &= \quad 1, \quad \sigma_i'^k, \quad \sigma_i'^k \cdot v_i', \quad q_i'^k, \quad [\Delta r' \times \sigma_i'^k] \\ R_{ji}^{(\psi)} &= \quad J_{ji}, \quad R_{ji}, \quad W_{ji}, \quad Q_{ji}, \quad D_{ji}. \end{align}$$ [7.9]

Let us introduce the average values $f_{ji}^{(\psi)}$ determined by the interaction between the carrier phase and one whole dispersed particle

$$f_{ji}^{(\psi)} = \frac{R_{jiV}}{n} = \frac{1}{dN_V} \sum_{\nu=1}^{dN_V} \int_{\zeta_{12}(\nu)} \psi_i'^k n_i'^k \, d's$$ [7.10]

where $\zeta_{12}(\nu)$ is the surface of the νth particle.

In view of the fact that the surfaces dS_{12S} and volumes dV_{iS} are in the vicinity of the boundary surface dS, we shall introduce macrovalues $(\varphi_i')_{jiS}^n$ and $(a_*\varphi_i')_{iS}^n$ related to the elementary macrosurface ds and depending on its orientation that is determined by an external unit normal $\mathbf{n}$

$$(\varphi_i')_{jiS}^n = \frac{1}{ds(\mathbf{n})} \int_{ds_{12S}} \varphi_i' \, d's$$ [7.11]

$$(a_*\varphi_i')_{iS}^n = \frac{1}{ds(\mathbf{n})} \int_{dv_{iS}} \varphi_i' \, d'V$$ [7.12]

where ds_{12S} and dv_{iS} are, respectively, the interface and volume of the ith phase in the cells adjacent to the cross-section ds and consisting of dn_S surfaces ζ_{12S} and volumes ϑ_{iS}. Since the elementary macrosurface ds intersects many of the cells, it is possible to assume that the averaged values introduced in [7.11] and [7.12] are stable or regular, i.e. as also $\langle\varphi_i'\rangle_i$ these are smoothly varying at distances $\sim L$. For values such as those of [7.11] and [7.12] at $\varphi_i' = \psi_i'^n$, $\varphi_i' = \nabla'^q\psi_i'^l$ and $\varphi_i' = \nabla'^l\psi_i'^l$, where ψ_i' is tensor or vector $(\psi_i'^n = \psi_i'^k n_i'^k)$, we shall introduce more compact designations

$$\psi_{jiS}^n = (\psi_i'^l n_i'^l)_{jiS}^n$$ [7.13]

$$(\psi_i'^{l,q})_{iS}^n = (a_*\nabla'^q\psi_i'^l)_{iS}^n$$ [7.14]

$$\psi_{iS}^n = (\psi_i'^l)_{iS}^n = (a_*\nabla'^l\psi_i'^l)_{iS}^n.$$ [7.15]

Later we shall have to do only with those cases, when ψ_{jiS}^n, $(\psi_i'^{l,q})_{iS}^n$ and ψ_{iS}^n will have the same tensor character and the same tensor dimension as the averaging values ψ_i'

$$\psi_{jiS}^n = \psi_{jiS}^k n^k$$ [7.16]

$$(\psi_i'^{l,q})_{iS}^n = (\psi_i'^{l,q})_{iS}^k n^k$$ [7.17]

$$\psi_{iS}^n = \psi_{iS}^k n^k \qquad [7.18]$$

where ψ_{jiS}^k, $(\psi_i^{l,q})_{iS}^k$ and ψ_{iS}^k correspond to three $(k = 1, 2, 3)$ directions of the interactions of ds^k with the normals along the Cartesian coordinate axes x^k. From [7.2], [7.3] by taking into account [7.16] and [7.17] one obtains

$$s_{12S}\langle\psi_i'^l n_i'^l\rangle_{12S} = R_{jiS}^{(\psi)} = \frac{1}{dV}\int_{dS_{12S}} \psi_i'^l n_i'^l \, d's = \frac{1}{dV}\oint_{dS} \psi_{jiS}^n \, ds = \nabla^k\psi_{jiS}^k \qquad [7.19]$$

$$\frac{dV_{iS}}{dV}\langle\nabla'^q\psi_i'^l\rangle_{iS} = \frac{1}{dV}\int_{dV_{iS}} \nabla'^q\psi_i'^l \, d'V = \frac{1}{dV}\oint_{dS} (\psi_i^{l,q})_{iS}^n \, ds = \nabla^k(\psi_i^{l,q})_{iS}^k. \qquad [7.20]$$

It is possible to obtain one more expression for $R_{12S}^{(\psi)}$ from [7.6] and [7.20]

$$R_{12S}^{(\psi)} = \nabla^k[\psi_{2S}^k - \alpha_2\langle\psi_2'^k\rangle_2] = \nabla^k\psi_{12S}^k. \qquad [7.21]$$

So, the determination of the averaged phase interactions at surfaces of the dS_{12S} type can be fulfilled by two methods. Note, that the presentation of $R_{21S}^{(\psi)}$ using the averaged value ψ_{21S}^k is determined by averaging microparameters of only the carrier phase (which will be used below) as distinct from the presentation in terms of $\langle\psi_2'^k\rangle_2$ and ψ_{2S}^k, the determination of which is associated with a parameter averaging within the dispersed phase.

Further on, we shall consider only such cases when at any instant the velocity fluctuations of phase transitions ξ_i' and $\xi_i'e_i'$ are not many times greater, as concerns an order of magnitudes, the corresponding average values. It is then possible to neglect the contribution of phase transitions at the dS_{12S} type surfaces

$$J_{12S} \ll J_{12V}, \quad J_{12S}e_{(12S)} \ll J_{12V}e_{(12V)i} \qquad [7.22]$$

$$R_{21S}^{(\psi)} \approx -R_{12S}^{(\psi)}. \qquad [7.23]$$

Hence, the action of the dispersed particles on the carrier phase can be represented in the form

$$R_{21}^{(\psi)} = nf_{21}^{(\psi)} + \nabla^k\psi_{21S}^k = nf_{21}^{(\psi)} + \nabla^k[\alpha_2\langle\psi_2'^k\rangle_2 - \psi_{2S}^k] \qquad [7.24]$$

and the action on the carrier and dispersed phases at the boundary dV along the surface $dS_i + dS_{jiS}$ is determined by the values

$$\langle\psi'^k\rangle - \psi_{2S}^k = \alpha_1\langle\psi_1'^k\rangle_1 + \psi_{21S}^k \quad (\langle\psi'^k\rangle = \alpha_1\langle\psi_1'^k\rangle_1 + \alpha_2\langle\psi_2'^k\rangle_2) \qquad [7.25]$$

$$\alpha_2\langle\psi_2'^k\rangle_2 - \psi_{21S}^k = \psi_{2S}^k. \qquad [7.26]$$

8. CONSERVATION EQUATIONS FOR DISPERSED MIXTURES

In accordance with [7.22]–[7.24] the equations of mass [3.9] and momentum [4.1], [4.2], as applied to the dispersed mixture take the form:

$$\frac{\partial\rho_1}{\partial t} + \nabla^k\rho_1 v_i^k = -nj \qquad [8.1]$$

$$\frac{\partial\rho_2}{\partial t} + \nabla^k\rho_2 v_2^k = nj \qquad [8.2]$$

$$\rho_1 \frac{d_1 v_1}{dt} = \nabla^k \sigma_{1*}^k + n f_{21} - nj(v_{(12)1} - v_1) + \rho g_1 \tag{8.3}$$

$$\rho_2 \frac{d_2 v_2}{dt} = \nabla^k \sigma_{2*}^k + n f_{12} + nj(v_{(12)2} - v_2) + \rho_2 g_2 \tag{8.4}$$

$$0 = f_{21} + f_{12} + j(v_{(12)2} - v_{(12)1}) \tag{8.5}$$

$$\sigma_{1*}^{kl} = \langle \sigma'^{kl} \rangle - \sigma_{2S}^{kl} + \Omega_1^{kl} = \sigma_1^{kl} + \sigma_{21S}^{kl} \tag{8.6}$$

$$\sigma_{2*}^{kl} = \sigma_{2S}^{kl} + \Omega_2^{kl}. \tag{8.7}$$

Here j and f_{ji} are the average intensity of phase transitions and an average force on the ith phase related to one dispersed particle (the values of the $f_{ji}^{(\psi)}$ type). The values σ_{i*}^{kl} will be referred to as the reduced stress tensors in phases, which could be regarded as average stress along $ds_{i*}^k = ds_i^k + ds_{21S}^k$. Even at the symmetric microstress tensor $\sigma_i'^{kl}$, the tensor σ_{1*}^{kl} can be asymmetric (for example, in case of intensive oriented particle rotation with the angular speed $\omega_2 \sim v_0/a$ (see [12.42]) due to σ_{2S}^{kl} or σ_{21S}^{kl}, i.e. due to incorporation in σ_{1*}^{kl} of the part of the interface force R_{21S} acting along ds_{21S}^k, although $\langle \sigma_i'^{kl} \rangle_i$, σ_i^{kl} and $\langle \sigma'^{kl} \rangle$ are symmetric.

Therefore, it is impossible to agree with Buevich & Markov (1973a) and Buevich & Shchelchkova (1976) assertions that the phenomenological introduction of the antisymmetric macrostresses in suspensions in the absence of antisymmetric stresses in microscale (as it was done by Afanasiev & Nikolaevsky 1969) has no physical sense. At the same time, one should realize that the presentation of the principal surface force vector with the asymmetric stress tensor σ_{1*}^{kl} in the form of $\nabla^k \sigma_{1*}^k + n f_{21}$ and with the symmetric stress tensor σ_1^{kl} in the form of $\nabla^k \sigma_1^k + R_{21}$ are identical, since $R_{21} = n f_{21} + \nabla^k \sigma_{21S}^k$, and neither of them is more general.

In Brenner's work (1970) it is affirmed that the introduction of the volume-average stress tensor is "unsatisfactory for a number of reasons": (1) There is no proof of the coincidence of volume-average and surface-average values, only the surface-average values being included in the equations. (2) In the case of a mixture where the particles are rigid bodies, the averaging is required in their interior, where the stresses are not uniquely defined. (3) In the case of interfacial (capillary) effects the volume-averaging leads to incorrect results (at the same time reference is made to an unpublished work and no mention is made as to what kind of incorrect results are meant). The first objection is removed as a result of the discussion of [2.7]. The second objection is also removed, since averaging within the particles $\langle \sigma_2'^{kl} \rangle_2$ is identically substituted for averaging of the carrier phase parameters at the surface of particles ds_{12S} and with the introduction of σ_{21S}^{kl}. As to the third objection, it is possible to extend these considerations to the case of capillary effects (see Nigmatulin 1978).

Let us also write out the equations of the phase fluctuating energy

$$\rho_1 \frac{d_1 k_1}{dt} = \nabla^k \Lambda_{1*}^k + \sigma_{1*}^{kl} \nabla^k v_1^l - \rho_1 A_1 + n(h_{21} - f_{21}^k v_i^k) - nj(k_{(12)1} - k_1) + \rho_1 H_1 \tag{8.8}$$

$$\rho_2 \frac{d_2 k_2}{dt} = \nabla^k \Lambda_{2*}^k - v_2^l \nabla^k \sigma_{2*}^{kl} - \rho_2 A_2 + n(h_{12} - f_{12}^k v_2^k) + nj(k_{(12)2} - k_2) + \rho_2 H_2 \tag{8.9}$$

$$0 = h_{21} + h_{12} + q_{21} + q_{12} + j(u_{(12)2} - u_{(12)1}) + 1/2 j(\langle v_2'^2 \rangle_2 - \langle v_1'^2 \rangle_1) \tag{8.10}$$

$$h_{ji} = f_{ji}^{(\psi)} \quad (\psi_i'^k = \sigma_i'^{kl} v_i'^l = c_i'^k) \tag{8.11}$$

$$q_{ji} = f_{ji}^{(\psi)} \quad (\psi_i'^k = q_i'^k) \tag{8.12}$$

$$\Lambda_{1*}^{k} = c_{1*}^{k} - \sigma_{1*}^{kl} v_{1}{}^{l} = \alpha_{1}[\langle \sigma_{1}^{\prime kl} \Delta v_{1}^{\prime l}\rangle_{1} - 1/2\langle \rho_{1}^{0\prime}(\Delta v_{1}^{\prime})^{2}\Delta v_{1}^{\prime k}\rangle_{1} + (\sigma_{1}^{\prime lq} n_{1}^{\prime q}\Delta v_{1}^{\prime ql})_{21S}^{k}] \qquad [8.13]$$

$$\Lambda_{2*}^{k} = (\nabla^{\prime q}(\sigma_{2}^{\prime ql} v_{2}^{\prime l}))_{2S}^{k} - 1/2\alpha_{2}\langle \rho_{2}^{0\prime}(v_{2}^{\prime})^{2}\Delta v_{2}^{\prime k}\rangle_{2}. \qquad [8.14]$$

Here Λ_{i*}^{k}, c_{i*}^{k} are the reduced fluxes (the values such as σ_{i*}^{kl}) of fluctuating energy and total mechanical energy; q_{ji}, h_{ji} are values of the $f_{ji}^{(\psi)}$ type (see [7.10]), i.e. these characterize the phase interaction per single particle; q_{ji} defining the heat flux into the ith phase, h_{ji} is the power of the work of surface forces in the ith phase.

If neither fast rotation, nor chaotic motion of dispersed particles, nor any other internal motion in their interior take place and particle collisions that lead to considerable fluctuations $\nabla^{\prime k}\sigma_{i}^{\prime kl}$ are absent too, then since the dispersed phase velocity fluctuations are small, it is possible to neglect momentum, energy and moment of the momentum of dispersed particles intersected by the boundary of the separated volume dV, in comparison with those for the particles entirely contained in the macrovolume. These assumptions of Buevich & Markov (1973a) considerably simplify the equations causing, in particular, the divergent terms at the right sides of the equations for the second (dispersed) phase to become zero and bring about coincidence of the reduced values of the stress tensor and energy fluxes in the carrier phase with the corresponding value for the mixture ($\langle\sigma^{\prime}\rangle^{kl} = \sigma_{1*}^{kl}$, $\sigma_{2*}^{kl} = 0$). As a result, the dispersed phase equations take the form of equations for a whole number of noninteracting particles. The case of oriented rotation of particles under the action of an external field ($M_{2} \neq 0$, $L_{2} \neq 0$) was considered by Brenner (1970b) and Buevich (1974).

Phenomenological theories of dispersed mixtures frequently correspond to a certain special case, when besides the above-mentioned, the following can be assumed:

$$\sigma_{1*}^{kl} = \langle\sigma^{\prime kl}\rangle = -p\delta^{kl} + \tau^{kl} \qquad [8.15]$$

$$nf = -\alpha_{2}\nabla p + nf_{*} \qquad [8.16]$$

$$c_{1*}^{k} = \langle c^{\prime k}\rangle = p(\alpha_{1}v_{1}{}^{k} + \alpha_{2}v_{2}{}^{k}) + \tau_{1}^{kl}v_{i}{}^{l} \qquad [8.17]$$

$$H_{i} = 0, \quad M_{1} = M_{2} = 0, \quad L_{1} = L_{2} = 0, \quad k_{1} = k_{2} = 0 \qquad [8.18]$$

that means that the fluctuating transfer of momentum and energy and also the momentum and energy of the fluctuating motion is ignored not only in the dispersed phase, but in the carrier phase too. Afanasiev & Nikolaevsky (1969) presented the phenomenological equations that take into consideration the oriented rotation of particles without the action of external field ($L_{2} = 0$).

9. A CELLULAR SCHEME FOR DISPERSED SYSTEMS

To determine the average values in the averaged equations, it is necessary to know the fields of velocity, stresses and the other parameters over the entire flow field. A detailed pattern of those fields can not, as a rule, be obtained due to complexity and variety of processes. It is, in fact, generally impossible to obtain this pattern even for the incomprassible viscous fluid about a single spherical particle and for different conditions we have to use different approximate schemes. The problem becomes especially difficult, when there is a plurality of particles, drops or bubbles, which can have a different shape, be deformed or fluctuate, disintegrate, stick together when phase transitions occurs at their surface, etc. Therefore, to obtain the rheologic relationships between the average parameters, it is necessary to resort to approximate schemes of micromotions about the dispersed particles. The "cellular" scheme or model used in the dispersed system theory and the kinetic theory of gases, is one possible element of such schemes.

Consider a monodispersed mixture, in any elementary macrovolume dV of which there are

spherical particles of the same radius a, and where, in accordance with the cellular scheme, some regular volume of the carrier phase (e.g. in the form of polyhedron or ball around a particle) corresponds, in average, to each dispersed particle. The motion inside this cell (distribution of velocities, densities, pressures and other parameters) is predetermined. The motion around the other dispersed particles of the elementary macrovolume is, in average, assumed to be the same as in the selected cell, i.e. it is supposed that some regular turbulence exists, as well as some "almost periodicity" of microparameters in the space with the linear period $2l$, equal to the average distance between impurities. On the basis of this model, by averaging over one cell, it is possible to derive the relations for the macroparameters contained in averaged equations.

The average volume of cell ϑ and the volume of phases ϑ_i in the cell are determined according to the abovementioned and [7.1] from the following relationships

$$\vartheta = \frac{dV_V}{dN_V} \approx \frac{dV}{dN} = \frac{1}{n}, \quad \vartheta_i = \frac{dV_{iV}}{dN_V} \approx \frac{dV_i}{dN} = \frac{\alpha_i}{n} \tag{9.1}$$

and the average values $\langle \varphi'_i \rangle_{iV}$ with respect to the volume consisting of a whole number of cells, and the average values $f_{ji}^{(\psi)}$ related to a single particle, we shall determine by integrating over a single particle (instead of using [7.2] and [7.10]):

$$\alpha_i = \frac{\vartheta_i}{\vartheta}, \quad s_{12} = \frac{\zeta_{12}}{\vartheta} \tag{9.2}$$

$$\langle \varphi'_i \rangle_{iV} = \frac{1}{\vartheta_i} \int_{\vartheta_i} \varphi'_i \, d'V \tag{9.3}$$

$$f_{ji}^{(\psi)} = \int_{\zeta_{12}} \psi'^k_i n'^k_i \, d's. \tag{9.4}$$

If chaotic motion of dispersed particles and their collisions are absent and the properties [9.2]–[9.4] are satisfied, then it is natural to refer to such motions of dispersed mixtures as laminar. In this case, though considerable fluctuations of microparameters do occur, these have a regular and periodic character.

Let us assume that the physical parameters of phases, like those of v'^k_i, σ'^{kl}_i and u'_i and so on, considerably vary within the boundaries of cell, their fluctuations do not exceed many times the corresponding average values. Then, in regard to the average values of physical parameters, it is possible to ignore the contribution of corresponding integrals with respect to the volume dV_{iS} (see [7.4]), i.e. it is possible to assume that

$$\langle \varphi'_i \rangle_i = \langle \varphi'_i \rangle_{iV}. \tag{9.5}$$

As distinct from the averaging over phases of the physical parameters themselves, when averaging their space derivatives over phases ($\nabla'^k v'^l_i$, $\nabla'^k \sigma'^{kl}_i$ and so on) and the values of the $\psi'^k_i n'^k_i (v'^k_i n'^k_i, \sigma'^{kl}_i n'^k_i$ and so on) type over the interface, it is necessary to keep in mind that the fluctuations of the mentioned values can be many times greater than the corresponding average values, as a result of it, the conditions may take place that lead to the necessity of taking into account the averages over the volume dV_{iS} and the interface dS_{12S}. From [7.4], [7.5] with [7.19], [7.20] we have

$$\langle \nabla'^q \psi'^l_i \rangle_i = \langle \nabla'^q \psi'^l_i \rangle_{iV} + \nabla^k (\psi'^{l,q}_i)^k_{iS} \tag{9.6}$$

$$s_{12} \langle \psi'^l_i n'^q_i \rangle_{12} = s_{12} \langle \psi'^l_i n'^q_i \rangle_{12V} + \nabla^k (\psi'^l_i n'^q_i)^k_{12S} \tag{9.7}$$

where, in accordance with [9.3], [9.4] it is possible to assume for the averages over the whole cells

$$\langle \nabla'^q \psi_i'^l \rangle_{iV} = \frac{1}{\vartheta} \int_{\vartheta_i} \nabla'^q \psi_i'^l \, \mathrm{d}'V \qquad [9.8]$$

$$s_{12} \langle \psi_i'^l n_i'^q \rangle_{12V} = \frac{1}{\vartheta} \int_{\zeta_{12}} \psi_i'^l n_i'^q \, \mathrm{d}'s. \qquad [9.9]$$

To select and substantiate the procedure of the determination of macroparameters $(\psi_i'^q)_{iS}^k$ and $(\psi_i'^l n_i'^q)_{12S}^k$, let us consider the cells in the vicinity of the elementary macroplane ds (see figure 3) having the external normal $\mathbf{n}$, which intersects dn_S cells, cutting off in each by the cross-section $\zeta(\nu) = \zeta_1(\nu) + \zeta_2(\nu)$ some volume of the ith phase ν_{iS} and certain particle surface $\zeta_{12S}(\nu)$. Then, according to [7.12] we have

$$(a_* \varphi')_{iS}^n = \frac{1}{\mathrm{d}s} \sum_{\nu=1}^{\mathrm{d}n_S} \int_{\vartheta_{iS}(\vartheta)} \varphi_i' \, \mathrm{d}'V$$

$$(\varphi_i')_{12S}^n = \frac{1}{\mathrm{d}s} \sum_{\nu=1}^{\mathrm{d}n_S} \int_{\zeta_{12S}} \varphi_i' \, \mathrm{d}'s \quad \mathrm{d}s = \sum_{\nu=1}^{\mathrm{d}n_S} \zeta(\vartheta).$$

When the centers of cells are spaced not very regularly, ds passes at different distances $x^n(\nu)$, counting from the cells centers along the normal $\mathbf{n}$. The values $x^n(\nu)$ define $\vartheta_{iS}(\nu)$, $\zeta_{12S}(\nu)$ and $\zeta(\nu)$ uniquely. By virtue of the inequality d$n_S \gg 1$ and the cyclic or repetitive nature of the microparameters from cell to cell, it is possible to replace the summation over dn_S cells by the integration with respect to x^n in one cell (see figure 3)

$$(a_* \varphi_i')_{iS}^n = \frac{1}{\mathrm{d}s} \frac{\mathrm{d}n_S}{2l} \int_{-l}^{l} \mathrm{d}x^n \int_{\vartheta_{iS}(x^n)} \varphi_i' \, \mathrm{d}'V \qquad [9.10]$$

$$(\varphi_i')_{12S}^n = \frac{1}{\mathrm{d}s} \frac{\mathrm{d}n_S}{2l} \int_{-l}^{l} \mathrm{d}x^n \int_{\zeta_{12S}(x^n)} \varphi_i' \, \mathrm{d}'s \qquad [9.11]$$

$$\mathrm{d}s = \frac{\mathrm{d}n_S}{2l} \int_{-l}^{l} \zeta(x^n) \, \mathrm{d}x^n = \frac{\mathrm{d}n_S}{2l} \vartheta \qquad [9.12]$$

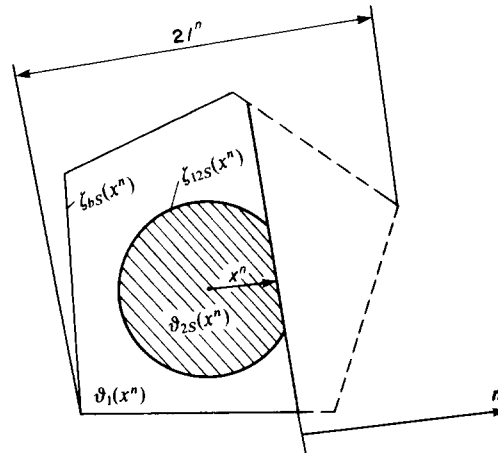Figure 3.

where l is the maximum radius of the cell in the n direction. It is possible to introduce in like manner the averages over the cell boundary ζ_b

$$s_b = \frac{\zeta_b}{\vartheta} \tag{9.13}$$

$$(\varphi_1')_{bV} = \frac{1}{\zeta_b} \int_{\zeta_b} \varphi_1' \, d's \tag{9.14}$$

$$(\varphi_1')_{bS} = \frac{1}{\vartheta} \int_{-l}^{l} dx^k \int_{\zeta_{bS}(x^k)} \varphi_1' \, d's. \tag{9.15}$$

In [3.8] the relationships [9.3], [9.6] are used for $\langle \nabla'^q \varphi_1'^l \rangle_1$ and the Gauss–Ostrogradsky formula is applied for the integral from $\nabla'^q \psi_i'^l$ with respect to the volume $\vartheta_{1S}(x^k)$ restricted by a particle surface $\zeta_{12S}(x^k)$ and by the cell cross-section $\zeta_1(x^k)$ occupied by the carrier phase, then, in view of [9.5] we obtain a condition that is equivalent to [3.8], but in the terms of parameter values at the cell boundary

$$s_b(\psi_1'^l n_1'^q)_{bV} + \nabla^k(\psi_1'^l n_1'^q)_{bS}^k = 0. \tag{9.16}$$

Let us assume a cell to be the spherical volume ϑ_R of the radius R, per one particle

$$\alpha_2 = \frac{\vartheta_2}{\vartheta}, \quad \vartheta_2 = \frac{4}{3}\pi a^3, \quad \vartheta = \vartheta_R = \frac{4}{3}\pi R^3 = \frac{1}{n} \tag{9.17}$$

$$s_{12} = \frac{3a^2}{R^3} = \frac{3\alpha_2}{a} \tag{9.18}$$

$$s_b = 3/R. \tag{9.19}$$

At the same time, we shall assume that the fluctuating or small-scale motion of the carrier phase encompasses only the spherical layer $\vartheta_{1c} = \{a < r < c\}$, where $a < c < R$, and outside this layer $(c < r < R)$ the disturbances are zero, considering this effect to be a result of the interference of neighbour particle influence. The value of the effective cell radius is characterized by the dimensionless parameter of cell

$$\eta_c = \frac{R^3}{c^3} \quad \left(1 \leqslant \eta_c \leqslant \frac{1}{\alpha_2}\right). \tag{9.20}$$

Note that in some cases it is possible to use step or discontinuous approximations of the *a priori* continuous distributions of the parameters in a cell. This is admissible, if these approximations are applied only when integrating over the phase volume in the cell.

A generalization of the aforementioned schemes is a more detailed (but also more complicated and ambiguous) scheme, which allows for a fixed form of cells (e.g. spherical) a certain set of cell dimensions R, and realization of each R is characterized by a probability $\varphi(R)$. Such scheme was used by Herchinsky & Pienkovska (1975) to determine the viscosity of concentrated suspensions.

10. FORMULATION OF THE PROBLEM ON THE MOTION OF FLUID ABOUT A DISPERSED PARTICLE

The closure of dispersed mixture macroscopic equations is associated with the analysis of microprocesses which occur in the vicinity of individual particles and resides in the deter-

mination of the distributions of displacements, velocities, temperatures, stresses, concentrations, etc. in a cell, which then should be integrated. The microanalysis is carried out independently and by a different method than the above procedure and is related with boundary-value problems in a single-phase continuum.

The micromotion of the carrier phase in a dispersed mixture is governed by its physical properties, conditions at the surface ζ_{12} of dispersed particles and at the cell boundary ζ_b.

We shall consider the case when the carrier phase may be described in terms of an incompressible Newtonian fluid with the density ρ_1^0 and viscosity μ_1. Then the equations and boundary conditions at the particle surface have the form

$$\nabla'^k v_1'^k = 0 \tag{10.1}$$

$$\rho_1^0 \frac{d_1 v_1'}{dt} = -\nabla' p_1' + \mu_1 (\nabla'^k \nabla'^k) v_1' + \rho_1^0 g_1' \tag{10.2}$$

$$r = a, \qquad v_1' = v_2 + [\omega_2 \times r] + w_{1a}\frac{r}{a}. \tag{10.3}$$

Any accurate predetermination of the boundary conditions at the cell boundary ζ_b is, generally speaking, impossible, since it would require solving of a certain problem encompassing all the dispersed particles, which is unrealistic. Therefore it seems to be expedient to resort to hypotheses that account for, in average, the "almost periodicity" of the dispersed system structure. These hypotheses are discussed, e.g. in the papers of Simha (1952), Safrai (1970) and Happel (1958).

We should now note the fact that v_1 is the average velocity of the carrier phase in the cell, which varies in small increments from cell to cell, and is regarded in macroequations as a continuous differentiable function due to spreading. The carrier phase velocity disturbances will be denoted here as $w = \Delta v_1' = v_1' - v_1$.

The nontranslational character of the carrier phase average motion is defined by the average velocity gradient tensor $v_1^{kl} = \nabla' v_1^k$, even in the general case of an incompressible carrier phase $v_1^{kk} = \text{div } v_1 \neq 0$, $v_1^{kl} \neq v_1^{lk}$ while the average velocity spatial variation in the vicinity of cell is determined by the equality $\Delta v_1^k = v_1^k(r) - v_1^k = v_1^{kl}x^l$, where $v_1(r)$ is the average velocity at an arbitrary point defined by the radius vector $r(x^1, x^2, x^3)$ relative to the center of cell in question. In this case, among possible assumptions there will be one, according to which at the cell boundary

$$r \in \zeta_b, \quad w^k = \Delta v_1'^k = \Delta v_1^k(r) = v_1^{kl}x^l. \tag{10.4}$$

At small volume concentrations of the dispersed phase α_2, somethimes it makes sense to somewhat weaken this condition by admitting derivations from it of an order α_2

$$r \in \zeta_b, \quad w^k = v^{kl}x^l(1 + 0(\alpha_2)). \tag{10.5}$$

As it was previously mentioned, it is possible to arbitrarily choose a cell form and conditions at its boundary ζ_b. These must always bring about such a motion in the cell, as will satisfy the precise conditions for the average velocity and the average gradient of velocity, the latter following from the second formula [3.8] as applied to

$$\langle v_1'^k \rangle_1 = v_1^k \tag{10.6}$$

$$\alpha_1 \langle \nabla'^l v_1'^k \rangle_1 = \nabla^l \alpha_1 v_1^k + s_{12}\langle v_1'^k n_1'^l \rangle_{12}. \tag{10.7}$$

.

Nobody has ever used the latter condition despite its apparency, and, as it was shown in [9.16], it is reduced for the cellular scheme to the equation for velocities at the cell boundary ζ_b

$$s_b(v_1'^k n_1'^l)_{bV} + \nabla^q(v_1'^k n_1'^l)_{bS}^q = 0. \tag{10.8}$$

The analytical solution of the problem [10.1]-[10.3], [10.7] is very complicated due to the finiteness of ϑ of a cell even in ultimate cases (an ideal fluid and extremely viscous fluid). For the sake of simplicity, in case of sufficiently small volume concentrations α_2 of the dispersed phase it is expedient to search for the solution in a cell as a part of the translational motion with the velocity v_∞^k (the fixed one in the cell) and the perturbed micromotion $w_{1\infty}^k$

$$v_1'^k = v_\infty^k + w_{1\infty}^k. \tag{10.9}$$

At the same time we shall assume that the field of v_1', apart from satisfying [10.1]-[10.3], is linearly asymptotic at its analytical continuation, i.e. it satisfies the linear boundary condition at infinity

$$r \longrightarrow \infty, \quad v_1'^k = v_\infty^k + v_\infty^{kl} x^l \tag{10.10}$$

defined by some, generally apeaking, different from v_1^{kl} velocity gradient tensor v_∞^{kl}. The values v_∞^k and v_∞^{kl} should be chosen so that the carrier phase momentum in the cell is defined by the velocity v_1^k and that the conditions [10.6], [10.7] are satisfied, which along with [10.8], [10.9] can be written in the form

$$v_1^k = v_\infty^k + \langle w_{1\infty}^k \rangle_{1V} \tag{10.11}$$

$$s_b \langle w_{1\infty}^k n_1'^l \rangle_{bV} + \nabla^q [(v_\infty^k + w_{1\infty}^k)n_1'^l]_{bS}^q = 0. \tag{10.12}$$

Note, that these conditions are sufficient to find the parameters of approximation v_∞^k and v_∞^{kl} through v_1^k and v_1^{kl}. Deviations from the conditions at the cell boundary ζ_b of the [10.4] type should be small, i.e. the conditions of the [10.5] type must be valid.

A further detailization of this scheme and its extension to the case of a greater α_2 may be attained, in particular, using a more complicated and general asymptotics than [10.10]

$$r \longrightarrow \infty, \quad v_1'^k = v_\infty^k + v_\infty^{kl} x^l + v_\infty^{klm} x^l x^m + v_\infty^{klmq} x^l x^m x^q + \ldots.$$

where the values v_∞^k, v_∞^{kl}, v_∞^{klm}, v_∞^{klmq}, etc. ought to be determined using the conditions for not only the average velocities $\langle v_1'^k \rangle_1$ and average first-order derivatives $\langle \nabla''v_1'^k \rangle_1$, but also for the average derivatives of a higher order $\langle \nabla''\nabla'^m v_1'^k \rangle_1$, $\langle \nabla''\nabla'^m \nabla'^q v_1'^k \rangle_1$ and so on, and by meeting the conditions at the boundary [10.4] with an accuracy greater than $0(\alpha_2)$.

By applying the continuity equation [10.1] to [10.10] when $r = \infty$, we have that the tensor v_∞^{kl}, unlike v_1^{kl}, has to satisfy the condition

$$v_\infty^{kk} = 0. \tag{10.13}$$

Apart from the absolute coordinate system where [10.1]-[10.3] are given, it makes sense to consider the "i"-coordinate systems moving with the average phase velocities v_i^k in the given cell. In these inertialess mobile coordinate systems the motion of carrier fluid is determined by the relative phase velocity v_{12}

$$v_{12} = v_1 - v_2 \tag{10.14}$$

and it is there necessary to account in the momentum equations for the force of inertia $\rho_i^0 v_i'$, equal in all points of the cell, besides the external mass forces $g_i' = g$), then the force of the action upon a particle from the carrier phase is equal to

$$f = f_{Ai} + f^*, \quad f_{Ai} = \rho_1^0 (v_i' - g) \vartheta_2 \quad (i = 1, 2), \quad f^* = f^*(\alpha_2, v_{12}, v_{12}')$$
[10.15]

where f^* is force defined by the relative phase motion v_{12} (in general, f^* can depend on all prehistory of $v_{12}(t)$), f_{Ai} is the so-called buoyancy force (see Batchelor 1970a and Sedov 1973) due to uniform inertial and mass forces.

Despite the evidence of [10.15], there are cases described in literature of the incorrect allowance for the inertial forces when analyzing a flow about dispersed particles.

In Fortier's book (1967) the component f_{Ai} was erroneously derived and used with the opposite sign because of v_2'.

In Buevich's & Markov's paper (1973b) the forces of inertia $\rho_i \, d_i v_i/dt$ are considered on the left sides of the momentum equations (see [4.4] of the cited paper and [4.1] of the present paper), but the inertial forces of the same order $\rho_1^0 \alpha_2 d_2 v_2/dt$ discussed here, which appear due to the force f_{A2} acting upon a sample dispersed particle, are not allowed for. In those cases when the left sides of said equations or the phase accelerations are essential, it makes [4.4] of Buevich's & Markov's paper (1973b) incorrect, independent of the form of $f^*(\alpha_2, v_{12})$. They are correct (see [12.53], [12.54] below) only when they are reduced to equilibrium equations ($\rho_i \, d_i v_i/dt \approx 0$).

11. THE MONODISPERSED NONCOLLISIONAL SYSTEM WHERE THE FLUCTUATING MOTION OF CARRIER PHASE IS APPROXIMATED BY A POTENTIAL FLOW OF AN IDEAL INCOMPRESSIBLE FLUID

Potential motion of an ideal ($\mu_1 = 0$) incompressible ($\rho_1^0 = $ const) fluid in a cell that satisfies [10.1]–[10.3], [10.10] (the boundary conditions at $r = a$ are set only to the normal velocity component: $v_1 \cdot r = v_2 \cdot r$) is described by the following formulas (see Nigmatulin 1978):

$$v_1'^k = \nabla^k \varphi = v_\infty^k + w_r^k + w_v^k + w_\nu^k$$
[11.1]

$$w_r^k = \frac{A}{r^3} x^k \quad (A(t) = w_{1a} a^2)$$
[11.2]

$$w_v^k = -\frac{B^k}{r^3} + \frac{3B^l}{r^5} x^l x^k \quad \left(B^k(t) = \frac{a^3(v_2^k - v_\infty^k)}{2} \right)$$
[11.3]

$$w_\nu^k = E^{kl} x^l - F^{lm} x^l x^m x^k$$
[11.4]

$$\left(E^{kl}(r, t) = v_\infty^{kl} \left(1 + \frac{2a^5}{r^5} \right), \quad F^{lm}(r, t) = \frac{5 v_\infty^{lm} a^5}{3 r^7} \right).$$

For the existence of such a potential motion, the potentiality of the velocity v_o field is necessary

$$v_0 = v_1 + 1/2 \alpha_2 v_{12} = \nabla \Phi.$$
[11.5]

From conditions [10.6], [10.7] we have expressions for the approximation parameters v_∞^k and v_∞^{kl}

$$v_\infty^k = v_1^k$$
[11.6]

$$\nu_\infty^{kl} = \nu_\infty^{lk} = 1/2(\nu_1^{kl} + \nu_1^{lk} - 2/3\nu_1^{mm}\delta^{kl}) - 1/20(\nabla^l\alpha_2 v_{12}^k + \nabla^k\alpha_2 v_{12}^l - 2/3\delta^{kl}\nabla^m\alpha_2 v_{12}^m)$$

$$= \nu_1^{kl} - 1/2\nu_1^{mm} + 0(\nabla^l\alpha_2 v_{12}^k). \tag{11.7}$$

The pressure p distribution in the cell is found from the Cauchy–Lagrange integral. Then it is possible to determine the force acting upon the particle:

$$f^k = \frac{4\pi a^3}{3}\rho_1^0\left[\frac{d_2 v_\infty^k}{dt} - g^k - \frac{1}{2a^3}\frac{d_2}{dt}(a^3(v_2^k - v_\infty^k)) - \frac{3}{2}(v_2^l - v_\infty^l)\nu_\infty^{lk}\right]$$

$$= \frac{4\pi a^3}{3}\rho_1^0\left[\frac{1}{2}\left(\frac{d_1 v_1^k}{dt} - \frac{d_2 v_2^k}{dt}\right) - \frac{v_{12}^k}{2}\left(\nabla^l v_1^l - \frac{3d_2 a}{a\,dt}\right) + \frac{d_1 v_1^k}{dt} - g^k\right]. \tag{11.8}$$

The first expression here coincides with the formula obtained by Voinov (1973) and Jakimov (1973). It is possible to express the force f only through average values and their derivatives owing to the relationships obtained [11.6], [11.7].

By integrating p' over the cell we can obtain the bubble radial motion equation that generalizes the Rayleigh–Lamb equation. It is possible to show by simple estimations that the nontranslational nature of v_1, which is characterized by the tensor ν_1^{kl} and is essential in f, gives a small contribution to the equation of radial motion. The latter equation may be obtained in another way starting from the kinetic energy equation for the fluctuating motion [8.8]. Since the approximation [11.1]–[11.4] in the cell involves a certain inaccuracy, the radial motion equations obtained by two above methods are somewhat contradictory. This contradiction may be removed to an accuracy of $\alpha_2^{1/3}$, if the parameters [9.20] of cell $\eta_{cv} = 1$, $\eta_{cr} = 3/2$ are used relating to velocity fields w_v and w_r, respectively. In case of not very strong ν_1^{kl}, the term due to the acceleration $\dot{w}_{1a}$ is the main term. Then, if it is assumed that $\eta_{cr} = 1.35$, $\eta_{cv} = 1$, the radial motion equations obtained by two methods can be regarded as the coinciding with an accuracy of $\sim\alpha_2$.

As a result, the system of equations for the phase mass, number of particles, phase momentum (if it is assumed that $v_{(12)1} \approx v_{(12)2}$), radial motion in a dispersed non-collisional mixture with the ideal incompressible carrier fluid takes the form (where p_{1a} is average fluid pressure on the bubble wall, Σ is the surface tension coefficient)

$$\frac{\partial\rho_1}{\partial t} + \nabla^k\rho_1 v_1^k = -nj, \quad \rho_1 = \rho_1^0\alpha_1, \quad j = 4\pi a^2\xi_{12} \tag{11.9}$$

$$\frac{\partial\rho_2}{\partial t} + \nabla^k\rho_2 v_2^k = nj, \quad \rho_2 = \rho_2^0\alpha_2 \tag{11.10}$$

$$\frac{\partial n}{\partial t} + \nabla^k n v_2^k = 0, \quad \alpha_2 = 4/3\pi a^3 n, \quad \alpha_1 = 1 - \alpha_2 \tag{11.11}$$

$$\frac{d_2 a}{dt} = w_{1a} + \frac{\xi_{12}}{\rho_1^0} = w_{2a} + \frac{\xi_{12}}{\rho_2^0}, \quad v_{(12)1} = v_{(12)2} = v_2 \tag{11.12}$$

$$\rho_1\frac{d_1 v_1}{dt} = \nabla^k\sigma_{1*}^k - nf - nj(v_2 - v_1) + \rho_1 g_1$$

$$= \alpha_1\nabla^k\sigma_{1*}^k - F - \alpha_1 nj(v_2 - v_1) + \rho_1 g_1 \tag{11.13}$$

$$\left(nf = \rho_1^0\alpha_2\left(\frac{d_1 v_1}{dt} - g\right) + nf_m = \alpha_2\nabla^k\sigma_{1*}^k + F, \quad F = \alpha_1 nf_m\right) \tag{11.14}$$

$$\rho_2\frac{d_2 v_2}{dt} = nf + \rho_2 g_2 = \alpha_2 \nabla^k \sigma_{1*}^k + F - \alpha_2 nj(v_2 - v_1) + \rho_2 g_2 \qquad [11.15]$$

$$\sigma_{1*}^{kl} = -p_{1*}\delta^{kl} + \tau_{1*}^{kl} \qquad [11.16]$$

$$\tau_{1*}^{kl} = -1/2\alpha_2\rho_1{}^0(v_{12}^k v_{12}^l - 1/3(v_{12})^2\delta^{kl}) \qquad [11.17]$$

$$f_m{}^k = \rho_1{}^0\frac{2\pi a^3}{3}\left[\frac{d_1 v_1{}^k}{dt} - \frac{d_2 v_2{}^k}{dt} + \frac{3 d_2 a}{a\,dt}v_{12}^k\right] \qquad [11.18]$$

$$(1 - \varphi_*^{(1)})a\frac{d_2 w_{1a}}{dt} = \frac{p_{1a} - p_{1*}}{\rho_1{}^0} - (1 - \varphi_*^{(2)})\frac{3 w_{1a}^2}{2} + (1 - \varphi_*^{(3)})\frac{v_{12}^2}{4} \qquad [11.19]$$

$$\varphi_*^{(1)} \approx 1.1\alpha_2^{1/3}, \quad \varphi_*^{(2)} \approx 1.47\alpha_2^{1/3} + 1.33\alpha_2, \quad \varphi_*^{(3)} \approx -0.67\alpha_2 \qquad [11.20]$$

$$p_2 = p_{1a} + 2\Sigma/a. \qquad [11.21]$$

For the given relationship $p_2(\rho_2{}^0)$ and the kinetics of phase transitions for ξ_{12}, the system [11.9]–[11.21] is closed.

In the work of Voinov & Petrov (1975) it is shown that coefficients in the addends of an order of $\alpha_2 w$ and $\alpha_2 v_{12}$ in the averaged equations of dispersed phase motion depend on a disposition of particle centers (see section 13 below), which can in principle vary. When a cellular scheme is used, this disposition is given beforehand according to the form and dimension of cell. This is the fact that defines the abovementioned coefficients.

The present cellular scheme, as also Garipov's analysis (1973) reveal the nonhydrostatic component of the stress tensor τ_{1*}^{kl} with the coefficient $K = -1/2$ (see [11.17]). As it was shown by Voinov & Petrov (1977), this result is accurate at $\alpha_2 \ll 1$ and the value of the coefficient K does not depend on the disposition of particles. Iordansky & Kulikowsky (1977) considered the case of nonspherical particles too. In Iordansky's (1960) work the coefficient $K = -1/20$ due to the disregard for the nonhydrostatic character of σ_{12S}^{kl} and the nontranslational character of averaged motion. The latter also leads to a formula for f_m that differs from [11.18]. In Jakimov's work (1973) the coefficient $K = 1$ that is related with two inaccuracies in the derivation of the formula for τ_{1*}^{kl}. The first of them consists in that it is assumed that

$$\left\langle \rho_1{}^0\frac{d_1' v_1'}{dt}\right\rangle_1 = \rho_1{}^0\frac{d_1 v_1}{dt}.$$

This is incorrect (see [3.12]) due to the fact that the addend $\alpha_1\langle\rho_1^{0'}\Delta v_1'^k\Delta v_1'^l\rangle_1$ of an order equal to the value of τ_{1*}^{kl} calculated here is not taken into consideration. The second discrepancy consists in that the fluid impulse change due to the departure from the volume V of one spherical dispersed particle is computed as impulse variation due to the displacement of the corresponding dipole center in an unbounded fluid. This is incorrect, since the perturbations $\Delta v_1'(\nu)$ initiated by the νth particle in the mixture consisting of many particles, are determined not only by the mentioned dipole term, but also by the disturbances "reflected" form the other particle surfaces (see Garipov 1973 and Voinov & Petrov 1975). These "reflected" disturbances are particularly tangible beyond the νth cell in the vicinity of the other "reflected" particles. In terms of the cellular scheme considered the presence of interference from all dispersed particles is allowed for by regarding the dipole and other terms defining $\Delta v_1'(\nu)$ as valid only inside the νth cell.

In the works referred to in this paragraph, the coefficients $\varphi_*^{(1)}$, $\varphi_*^{(2)}$, $\varphi_*^{(3)}$ characterizing the nonsingularity of bubbles in [11.19] have not been taken into account.

12. THE MONODISPERSED NONCOLLISIONAL SYSTEM, WHERE THE FLUCTUATING MOTION OF CARRIER PHASE IS APPROXIMATED BY THE STOKES OR CREEPING FLOW OF AN INCOMPRESSIBLE VISCOUS FLUID

Another extreme case is that of low Reynolds number micromotion ($Re = \rho_1^0 w_0 a/\mu_1 \ll 1$, where $w_0 = \max(w_{1a}, v_{12}, \omega_2 a)$), when the effect of the nonlinear internal forces in [10.2] is small. Let us first consider the case when the linear inertial forces are also small due to nonstationarity ($\rho_1^0 w_0/t_0 \ll \mu_1 w_0/a^2$, where t_0 is the characteristic time of variations in the fluid motion relative to a particle). Using to some extent different method than that considered below, the present case has been discussed in the papers of Brenner (1970a) and Buevich & Markov (1973). Determination of the motion in a cell is reduced to the solution of simultaneous linear Stokes equations with [10.10] which are represented as the superposition of the creeping flows known in literature (see Happel & Brenner 1965):

$$v_1' = v_2 + w_v + w_e + w_\omega + w_r \tag{12.1}$$

$$w_v^k = v_\infty^k - v_2^k - \frac{1}{4}\left(\frac{3a}{r} + \frac{a^3}{r^3}\right)(v_\infty^k - v_2^k) - \frac{3}{4}\left(\frac{a}{r^3} - \frac{a^3}{r^5}\right)(v_\infty^m - v_2^m)x^m x^k \tag{12.2}$$

$$w_e^k = \left(1 - \frac{a^5}{r^5}\right)e_\infty^{km}x^m + \frac{5}{2}\left(\frac{a^5}{r^7} - \frac{a^3}{r^5}\right)e_\infty^{mi}x^m x^i x^k \tag{12.3}$$

$$w_\omega^k = \omega_\infty^{km}x^m - \frac{a^3}{r^3}(\omega_\infty^{km} - \omega_2^{km})x^m \tag{12.4}$$

$$w_r^k = \frac{a^2 w_{1a}}{r^3}x^k \tag{12.5}$$

$$p_1' = p_\infty + \rho_1^0\left(g^m - \frac{d_2 v_2^m}{dt}\right)x^m - \frac{3a\mu_1}{2r^3}(v_\infty^m - v_2^m)x^m - \frac{5a^3\mu_1}{r^5}e_\infty^{mi}x^m x^i \tag{12.6}$$

$$e_\infty^{km} = 1/2(v_\infty^{km} + v_\infty^{mk}), \quad \omega_\infty^{km} = 1/2(v_\infty^{km} - v_\infty^{mk}) \tag{12.7}$$

$$\omega_2^{km} = \epsilon^{kml}\omega_2^l. \tag{12.8}$$

Here ϵ^{kml} is the Levi–Chivitta tensor. By employing [10.11], [10.12] we shall obtain

$$v_\infty = v_1 + \frac{\beta_2}{1-\beta_2}(v_1 - v_2) \quad \left(\beta_2 = \frac{3}{2}\frac{\alpha_2^{1/3} - \alpha_2}{\alpha_1}\right) \tag{12.9}$$

$$p_\infty = p_1 = \langle p_1'\rangle_1 \tag{12.10}$$

$$\alpha_1 e_\infty^{kl} = e^{kl} - 1/3\, e^{mm}\delta^{kl} + \Delta\, e_{12}^{kl} \tag{12.11}$$

$$\alpha_1 \omega_\infty^{kl} = \omega^{kl} - \alpha_2\omega_2^{kl} + \Delta\omega_{12}^{kl} \tag{12.12}$$

where e^{kl} is strain rate tensor, ω^{kl} is rotation tensor, which are defined by the volume-average mixture velocity v

$$e^{kl} = 1/2(\nabla^k v^l + \nabla^l v^k) \tag{12.13}$$

$$\omega^{kl} = 1/2(\nabla^k v^l - \nabla^l v^k) \tag{12.14}$$

$$\Delta e_{12}^{kl} \sim \Delta \omega_{12}^{kl} \sim \nabla^k(\alpha_2^{1/3} v_{12}^l)$$ [12.15]

$$v = \alpha_1 v_1 + \alpha_2 v_2.$$ [12.16]

The formulas for the force f, momentum d, stress tensors σ_{1*}^{kl} and $\langle \sigma'^{kl} \rangle$, the power of the work of internal viscous forces A_1 take the form

$$f = f_{A2} + f_S$$ [12.17]

$$nf_{A2} = \rho_1^0 \alpha_2 \left(\frac{d_2 v_2}{dt} - g \right)$$ [12.18]

$$nf_S = \frac{9\mu_1 \alpha_2 \varphi_f}{2a^2}(v_1 - v_2)$$ [12.19]

$$nd = 6\mu_1 \alpha_2 \alpha_1^{-1}(\omega - \omega_2 - \Delta \omega_{12})$$ [12.20]

$$\sigma_{1*}^{kl} = \langle \sigma' \rangle^{kl} - \sigma_{2S}^{kl}$$ [12.21]

$$\langle \sigma' \rangle^{kl} = -p_1 \delta^{kl} + \zeta_{(\sigma)} e^{mm} \delta^{kl} + 2\mu_{(\sigma)}(e^{kl} - 1/3\, e^{mm}\delta^{kl}) + \mu_{12}\Delta\, e_{12}^{kl} + 1/5\rho_2 a^2 \omega_2^{km} \omega_2^{lm}$$ [12.22]

$$\sigma_{2S}^{kl} = \kappa(\omega^{kl} - \omega_2^{kl} - \Delta\omega_{12}^{lk}) + 1/5\rho_2 a^2 \omega_2^{km} \omega_2^{lm}$$ [12.23]

$$(\zeta_{(\sigma)} = -4/3\mu_1, \quad \kappa = 3\mu_1 \alpha_2 \alpha_1^{-1}$$ [12.24], [12.25]

$$\mu_{(\sigma)} = \mu_1 \alpha_1^{-1}(1 + 3/2\alpha_2), \quad \mu_{12} = 5/2\mu_1 \alpha_2 \alpha_1^{-1})$$ [12.26], [12.27]

$$\rho_1 A_1 = 2\mu_1 \varphi_A^{(1)}\left(e^{kl} - \frac{e^{mm}}{3}\delta^{kl} \right) + \frac{4\mu_1 \alpha_1}{\alpha_2}e^{kk}\, e^{ll} + \frac{9\mu_1 \alpha_2 \varphi_A^{(2)}}{a^2}v_{12}^k v_{12}^k + 6\mu_1 \alpha_2 \alpha_1^{-1}(\omega^k - \omega_2^k)(\omega^k - \omega_2^k)$$

$$+ \mu_1 0(e^{kl}\, \Delta e_{12}^{kl}, \alpha_2(\omega^{kl} - \omega_2^{kl})\, \Delta\omega_{12}^{kl})$$ [12.28]

$$\varphi_A^{(1)} = \frac{1}{\alpha_1}\left(1 + \frac{3\alpha_2}{2\alpha_1} - \frac{15\alpha_2^2}{2\alpha_1} + 0(\alpha_2^{8/3}) \right) = 1 + 5/2\alpha_2 + 0(\alpha_2^2)$$ [12.29]

$$\varphi_A^{(2)} = 1 + 3/2\alpha_2^{1/3} + 9/4\alpha_2^{2/3} + 11/8\alpha_2 + 0(\alpha_2^{4/3})$$ [12.30]

$$\varphi_f = (1 - \beta_2)^{-1} = 1 + 3/2\alpha_2^{1/3} + 9/4\alpha_2^{2/3} + 15/8\alpha_2 + 0(\alpha_2^{4/3}).$$ [12.31]

In the case of low-concentrated suspensions and in the absence of a translational, rotational and radial motion in the vicinity of particles, i.e. when the conditions are satisfied

$$\alpha_2^{4/3} \ll 1$$ [12.32]

$$v_2 = v_1 = v, \quad \omega_2 = \omega$$ [12.33], [12.34]

$$w_{1a} = 0 \quad (e^{kk} = 0)$$ [12.35]

such a dispersed mixture may be regarded as the incompressible Newtonian fluid

$$\sigma^{kl} = \sigma^{lk} = -p\delta^{kl} + \tau^{kl}, \quad \tau^{kl} = \tau^{lk} = 2\mu\, e^{kl}$$ [12.36]

$$\rho A = \tau^{kl}\, \mathrm{e}^{kl} \qquad\qquad\qquad [12.37]$$

$$\mu = \mu_1(1 + 5/2\alpha_2), \quad \rho = \rho_1{}^0(1 - \alpha_2) + \rho_2{}^0\alpha_2. \qquad\qquad [12.38]$$

In this case from [12.11] it follows that

$$\mathrm{e}_\infty^{kl} = \frac{\mathrm{e}^{kl}}{\alpha_1}. \qquad\qquad\qquad [12.39]$$

This condition is in fact the corollary of the Gauss–Ostrogradsky theorem for the integral over the cell volume from $\nabla^k v_1{}^l$. The disregarding of [12.39] caused Pokrovsky (1968) to make an inaccuracy in that he took $\mathrm{e}_\infty^{kl} = \mathrm{e}^{kl}$ and on the basis of the correct relationship

$$\langle\sigma'\rangle^{kl} = -p\delta^{kl} + 2\mu_1(1 + 3/2\alpha_2)\mathrm{e}_\infty^{kl} \qquad\qquad [12.40]$$

came to the conclusion that Einstein's formula should be amended by assuming $\mu = \mu_1(1 + 3/2\alpha_2)$.

In case of an incompressible carrier phase, when $v_1 = v_2$, the compressibility of the mixture ($\mathrm{e}^{kk} = 3\alpha_2 w_{1a}/a$) can be manifested only in the radial motion ($w_{1a} \neq 0$) and the mixture becomes a non-Newtonian one

$$\zeta_{(\sigma)} = -\frac{4}{3}\mu_1 < 0, \quad \zeta_{(A)} = \frac{4\mu_1\alpha_1}{\alpha_2} > 0 \quad (\zeta_{(\sigma)} \neq \zeta_{(A)}). \qquad [12.41]$$

In the interesting analysis of Brenner (1970a) devoted to the case in question, the fact was not regarded that when $\mathrm{e}^{kk} \neq 0$ (though $\mathrm{e}_\infty^{kk} = 0$) radial motion should exist. In this cited paper in place of the boundary condition [10.3] at the particle surface $r = a$ the condition $v_1' = v_2 + [\omega_2 \times r]$ was used that is only valid for $\mathrm{e}^{kk} = 0$. It led to the incorrect value of volumetric viscosity ($\zeta_{(\sigma)} = \zeta_{(A)} = (4/3)\mu_1\alpha_2$).

In presence of the oriented rotation of particles that can be sustained by virtue of their inertia after the effect of the discontinuity of external moments, the reduced stress tensor σ_{1*}^{kl} is asymmetric, though σ_1^{kl} and $\langle\sigma'\rangle^{kl}$ are symmetric (see the remark after [8.7] and also the end of section 8)

$$\epsilon^{ikl}\sigma_{1*}^{kl} = 2\kappa(\omega^i - \omega_2{}^i - \Delta\omega_{12}^i) = n\, \mathrm{d}^i \qquad\qquad [12.42]$$

$$(\kappa = 3\mu_1\alpha_2/\alpha_1).$$

The consideration of the inertial forces $\rho_1{}^0\partial v_1'/\partial t$ due to the nonstationarity (when $Re \ll 1$ is preserved) is analytically fulfilled (see, e.g. Fortier 1967) for the rectilinear particle motion too. In this case the force f depends on the prehistory and at a certain approximation has the form:

$$f = f_{A1} + f^* \qquad\qquad\qquad [12.43]$$

$$f^* = f_m + f_S + f_B \qquad\qquad\qquad [12.44]$$

$$f_{A1} = \frac{4\pi a^3}{2}\rho_1{}^0\left(\frac{\mathrm{d}_1 v_1}{\mathrm{d}t} - g\right) \qquad\qquad [12.45]$$

$$f_m = \frac{2\pi a^3}{3}\rho_1{}^0\left(\frac{\mathrm{d}_1 v_1}{\mathrm{d}t} - \frac{\mathrm{d}_2 v_2}{\mathrm{d}t}\right) \qquad\qquad [12.46]$$

$$f_S = \frac{4\pi a^3}{3}\frac{9\mu_1}{2a^2}v_{12}$$

[12.47]

$$f_B = \frac{4\pi a^3}{3}\frac{9}{2a}\left(\frac{\rho_1^0\mu_1}{\pi}\right)^{1/2}\int_0^t \frac{d_2 v_{12}}{dt}\frac{d\tau}{\sqrt{(t-\tau)}}.$$

[12.48]

The effect of nonstationarity is defined by the dimensionless parameter

$$K = 2a\left(\frac{\rho_1^0}{\mu_1 t_0}\right)^{1/2} = \sqrt{(St \cdot Re)} \qquad \left(St = \frac{2a}{w_0 t_0}\right).$$

[12.49]

At the same time, in the extreme cases

$$K \ll 1 \quad (Re \ll 1), \qquad f_m \ll f_B \ll f_S$$

$$K \gg 1 \quad (Re \ll 1), \qquad f_m \gg f_B \gg f_S$$

it is possible to neglect the inherited Basset force f_B.

Kinematic equations for the dispersed mixture in question have the same form as in the preceeding case (see [11.9]–[11.12]). The momentum equations also have a similar form

$$\rho_1 \frac{d_1 v_1}{dt} = \nabla^k \sigma^k_{1*} - nf - nj(v_2 - v_1) + \rho_1 g_1 =$$

$$= \alpha_1 \nabla^k \sigma^k_{1*} - F - \alpha_1 nj(v_2 - v_1) + \rho_1 g_1$$

[12.50]

$$\rho_2 \frac{d_2 v_2}{dt} = nf + \rho_2 g_2 =$$

$$= \alpha_2 \nabla^k \sigma^k_{1*} + F - \alpha_2 nj(v_2 - v_1) + \rho_2 g_2$$

[12.51]

$$(nf = nf_{A1} + nf^* = \alpha_2 \nabla^k \sigma^k_* + F, \quad F = \alpha_1 nf^*).$$

[12.52]

In the cases when it is possible to neglect inertial forces, these equations transform into the equilibrium equations of external mass forces, buoyancy and Stokes forces, which in the case of $g_1 = g_2 = g$ can be written in the following form:

$$\nabla^k \sigma^k_{1*} + (\rho_1 + \rho_2)g = 0$$

[12.53]

$$9/2\mu_1 a^{-2}\varphi_f \alpha_2(v_1 - v_2) + (\rho_2^0 - \rho_1^0)g = 0.$$

[12.54]

13. DIFFERENT APPROXIMATIONS FOR ALLOWANCE OF THE INFLUENCE OF THE VOLUME CONCENTRATION OF DISPERSED PARTICLES ON THE VALUES DEFINING INTERACTIONS IN A DISPERSED MIXTURE

The coefficients $\varphi_*^{(1)}$, $\varphi_*^{(2)}$, $\varphi_*^{(3)}$ in [11.19] for radial oscillations of a bubble and the coefficients φ_f, $\varphi_A^{(1)}$, $\varphi_A^{(2)}$ in expressions [12.19] and [12.28] for the viscous Stokes force f_S and the work of internal forces A_1 characterize the nonsingularity of particles and their reciprocal influence on a flow about them. Within the frame of the model with equal cells considered, corresponding to the uniform distribution of the dispersed particles with fixed distances

between their centers, the effect of the finiteness of α_2 is determined by the magnitude of $\sim \alpha_2^{1/3}$ (see [11.20], [12.29]–[12.31])

$$\varphi_* = 1 - O(\alpha_2^{1/3}) \quad (\varphi_* = \varphi_*^{(1)}, \varphi_*^{(2)}) \tag{13.1}$$

$$\varphi = 1 + O(\alpha_2^{1/3}) \quad (\varphi = \varphi_f, \varphi_A^{(2)}). \tag{13.2}$$

By measuring the velocity $v_2 = v_g$ of the stationary sedimentation of dispersed particles due to gravity $(g = 9.81 \text{ m/s}^2)$ in a mixture at rest with the viscous carrier fluid, we have from [12.54]:

$$9/2\mu_1 a^{-2}\varphi_f(v_2 - v_1) = (\rho_2^0 - \rho_1^0)g = 9/2\mu_1 a^{-2}v_{g0} \tag{13.3}$$

$$\alpha_1 v_1 + \alpha_2 v_2 = 0. \tag{13.4}$$

It is easy to derive $\varphi_f(\alpha_2)$

$$\varphi_f = \frac{\alpha_1 v_{g0}}{v_g} \quad \left(\frac{v_g}{v_{g0}} = \frac{1 - \alpha_2}{\varphi_f}\right) \tag{13.5}$$

where v_{g0} is a single particle settling velocity $(\alpha_2 \approx 0)$.

The experimental data for $0 \leq \alpha_2 \leq 0.05$ given in the book of Happel & Brenner (1965) show that, despite $Re \ll 1$, at least three types of relationships

(1) $$v_g/v_{g0} = 1 - K^{(1)}\alpha_2^{1/3}, \quad K^{(1)} \approx 1.3 \div 1.8 \tag{13.6}$$

(2) $$v_g/v_{g0} = 1 - K^{(2)}\alpha_2, \quad K^{(2)} \approx 5 \div 6 \tag{13.7}$$

(3) $$v_g/v_{g0} = 1 + K^{(3)}(\alpha_2), \quad K^{(3)} \approx 0 \div 0.6 \tag{13.8}$$

are realized for different fluids and spherical particles. Relationship [13.6] conforms to φ_f obtained from the cellular model considered for uniformly distributed particles. Relationship [13.8] is valid for considerable agglomeration of particles (since the aggregates settle faster than the same quantity of uniformly distributed particles) and for arrangement of particles in line one after another. Relationship [13.7] is an intermediate one and can be possibly realized when the particles are distributed without agglomeration, but, at the same time, nonuniformly, i.e. randomly and chaotically, when the neighbour particles can be disposed very close (in particular, can touch one another), and also when they are rather distant.

It is extremely difficult to solve the problem describing a flow about a system of arbitrarily spaced particles even for the limiting linear formulations of a creeping flow of viscous fluid and a potential flow of an ideal fluid. Recently, a number of investigators have used an approximate method that allows in the cases of the aforementioned limited linear formulations for not extremely large concentrations of dispersed phase to follow up the effect of the disposition of dispersed particles on the interaction in a dispersed mixture. For this purpose the following fact is used that under the above formulations, a flow of a carrier fluid about one particle can be represented as a result of the action of a certain point singularity (source, sink, dipole, force, etc.) in the fluid occupying the infinite region $0 < r < \infty$. In this relation, a flow about a sample particle with the center at $r = 0$ in the dispersed mixture can be considered as the flow of liquid occupying all the flow field $a < r < \infty$, where in place of the remaining (secondary) particles, the prescribed point singularities act, which are spaced into the points corresponding to the centers $r = r^{(i)}$ $(i = 1, 2, .., N; r^{(i)} > 2a)$ of the secondary particles. The mentioned singularities (usually

they are the δ-singularities) are considered in the form of point forces and point sinks in the differential equations of momentum and continuity defining the field of velocities v_1'.

Disposition of the secondary particle centers or the structure of dispersed mixture are taken into account with the help of a distribution function P that indicates the probability of the disposition of $(r^{(1)}, r^{(2)}, \ldots, r^{(N)})$ and makes it possible to introduce average-over-ensemble values. In particular, the average carrier fluid velocity is defined as

$$v_1(r) = \int \cdots \int_{r^{(i)}>2a} v_1'(r)P(r^{(1)}, r^{(2)}, \ldots, r^{(N)}) \, dr^{(1)} \, dr^{(2)} \ldots dr^{(N)}. \qquad [13.9]$$

The distribution function introduced and average-over-ensemble values are defined by the binary distribution function $P_*(r)$ that shows a probability of location of the secondary particle center in the neighborhood of the end of r. This function is assumed spherically symmetrical in the form of $P_*(r)$. Starting from the definition of the number concentration of dispersed particles n we have the normalization condition:

$$\lim_{R \to \infty} \frac{1}{4/3\pi R^3} \int_{2a}^{R} 4\pi r^2 P_*(r) \, dr = n. \qquad [13.10]$$

For case of the chaotic disposition of the secondary particle centers, if we neglect their nonpoint reciprical character, it is possible to take (see Line 1 in figure 4):

$$P_*(r) = \begin{cases} n, & r > 2a \\ 0, & r < 2a \end{cases}. \qquad [13.10]$$

A more accurate analysis of the chaotic distribution of the secondary particles with allowance for their nonpoint character, because of which the secondary particle centers can not be close to one another more than $2a$ ($|r^{(i)} - r^{(j)}| \geq 2a$), leads to a curve of the 2 type in figure 4. At $\alpha_2 \ll 1$ the nonpoint character of the secondary particles is insignificant, and the difference between curves 1 and 2 is small. But one should bear in mind that the nonpoint character of the secondary particles is counted not only in $P_*(r)$.

Further on, by means of the distribution function or binary correlative function $P_*(r)$ the momentum and continuity equations are averaged. As a result of averaging of the point singularities, the nonzeroth terms for $r \geq 2a$ are obtained, which can be considered in the
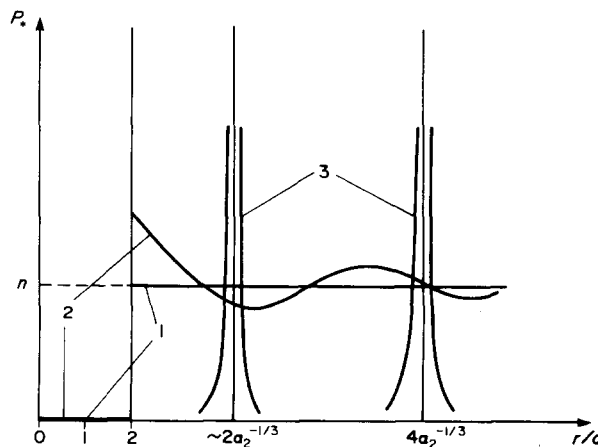

Figure 4.

averaged momentum equation as distributed volume forces due to the presence of the secondary particles. As a result of the solution of the averaged equations (and for the case of creeping motion they take the form of the Stokes equations with an additional nonsingular, but discontinuous addend at $r = 2a$), it is possible to determine the averaged action upon a sample particle. The proximity of this method is in the disregard for the nonpoint character of the secondary particles, the boundary conditions at their surfaces and with allowance only in average for the boundary conditions at the surface of a sample particle.

Using the method described, Golovin & Chizhov (1978) have obtained or proved a relationship of the [13.7] type. In the same work this analysis was generalized to the case of moderate concentrations ($\alpha_2 \lesssim 0.2$). At the same time $P_*(r)$ was used that allows for the nonpoint character of the secondary particles (Line 2 in figure 4). As a result, the formula was obtained

$$v_g/v_{g0} = 1 - 5\alpha_2 + 10\alpha_2{}^2. \qquad [13.11]$$

The formulas of the [13.7] type for the case of small α_2 was obtained by Batchelor (1972) and Saffman (1973) by means of a somewhat different method.

Kroshilin, Kroshilin and Nigmatulin, by using [13.10] and the approximation of thin boundary layer about a particle and a potential flow of ideal fluid outside it (that is valid for the description of a flow about spherical bubbles at $Re_v \gg 1$), have obtained

$$\varphi_f = 1 + \alpha_2 \quad \text{or} \quad v_g/v_{g0} = 1 - 2\alpha_2 \qquad [13.12]$$

that is in a good agreement with the experimental data on the lift of an ensemble of bubbles.

The same authors have considered by using [13.10] the effect of the nonindividuality of bubbles (at the chaotic distribution of the distances between them) in the Rayleigh–Lamb equation for radial oscillations of bubbles and have obtained

$$\varphi_*{}^{(1)} = 1 - 4.5\alpha_2, \qquad \varphi_*{}^{(2)} = 1 - 12\alpha_2 \qquad [13.13]$$

Additionally, it was shown that the use of $P_*(r)$ with the δ-singularities at distances multiple to a twofold cell radius $a\alpha_2^{-1/3}$ (see Curve 3 in figure 4) led to the same results as also the cellular scheme. So, the validity of this approximate approach is in the fact that it permits simulation of different dispositions of particle centers and to obtain the effect of this disposition on the coefficients taking into account constraint.

In the work of Buevich et al. (1977) an approach was used that was slightly different from that described above, where a creeping flow about a sample particle was considered as the flow of uniform "averaged" liquid with the effective viscosity of mixture μ and the average density ρ about it. In the case of suspension settling, it leads to the following equilibrium equation

$$\alpha_2\rho_2{}^0 g - \alpha_2\rho^0 g + 9/2\alpha_2\mu a^{-2}(v - v_g) = 0 \qquad [13.14]$$

$$\mu = \mu_1(1 + 5/2\alpha_2), \quad \rho = \rho_1{}^0\alpha_1 + \rho_2{}^0\alpha_2. \qquad [13.15]$$

Hence with the allowance for [13.4]

$$\frac{v_g}{v_{g0}} = \alpha_1\frac{\mu}{\mu_1} \approx 1 - 7/2\alpha_2 \qquad [13.16]$$

is obtained that although it differs from [13.7] and the results marked previously for the chaotic structure of monodispersive mixture, can be considered as one of possible formulas for the

description of dispersed mixtures with a chaotic distribution of particle centers in an extremely viscous fluid due to the multiple value of experimental data at hand and the insufficient substantiation of the modern theoretical approaches.

We have one more approach that was suggested by Brinkman and was discussed by Tam (1969), where in the analysis of a creeping flow of viscous fluid about a sample particle the effect of the secondary particles is taken into consideration by the drag force, additionally distributed in the liquid and proportional to $v_1' - v_2$. As a result, a correction in the force is obtained as intermediate between [13.6] and [13.7]

$$v_g/v_{g0} = \alpha_1/\varphi_f = 1 - 3/2\sqrt{(2\alpha_2)}.$$ [13.16]

Apart from the abovementioned, there are a number of phenomenological or empirical methods to account the effect of volume concentration α_2 of the dispersed phase on the force f per one particle for moderate values of α_2.

In conclusion we shall note one interesting fact. The corrections of an order of α_2 or $\alpha_2^{1/3}$ to the nonindividuality of particles in the expression for the viscous friction force f_s and in the equation for radial bubble fluctuations depend on a structure of the disposition of particles in a mixture. At the same time the correction $(5/2)\alpha_2$ at the coefficient of mixture viscosity (see [12.38]) and the correction τ_{1*}^{kl} (see [11.17]) that define the shear parts of stress tensor, do not depend on the mentioned structure.

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